Perfectoid geometry of p-adic modular forms

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Abstract

We prove a perfectoid tilting isomorphism that describes the Hecke module of overconvergent t-adic modular forms of Andreatta–Iovita–Pilloni at the boundary of weight space in terms of p-th power sequences of overconvergent p-adic modular forms of weights converging to the boundary. This isomorphism relies on a theory of perfectoid modular forms, which we define using Scholze's modular curves of infinite level.

For the proof, we study p-adic families of perfectoid modular forms over a perfected p-adic weight space, as well as equicharacteristic families of t-adic modular forms. Our results give a close analogy between perfectoid modular forms and perfectoid algebras: We prove that integral sheaves of perfectoid modular forms are almost acyclic, and construct an analogue of the canonical lift and the $\sharp$-map, which canonically extend a t-adic modular form into a family over a large weight space annulus.

Finally, we give some conjectural relations to Coleman's Spectral Halo conjecture.

Contents

1 Introduction

1.1 Context: The eigencurve at the boundary of weight space

Let p be a prime and let $N \geq 3$ be an integer coprime to p. Let X be the compactified modular curve of level $\Gamma(N)$ over $\mathbb{Z}_p$. Let $\mathfrak{X}$ be its p-adic completion, considered as a p-adic formal scheme. For any p-adically complete $\mathbb{Z}_p$-algebra R, we shall denote by $\mathfrak{X}_R := \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(R)$ the base change to R.

The context of this work is in the theory of p-adic modular forms, as first constructed by Serre [54] by considering p-adic limits of q-expansions of modular forms over $\mathbb{Q}$. Serre realised that in order to define weights for such limits, one would have to pass from weights $k \in \mathbb{Z}$ to continuous group homomorphisms $\kappa : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$, called p-adic weights. Here the former are embedded into the latter by sending k to the homomorphism $x \mapsto x^k$. Katz [34] reinterpreted Serre's work in geometric terms shortly after, by constructing a line bundle $\mathfrak{w}^\kappa$ of modular forms for any complete $\mathbb{Z}_p$-algebra R and any continuous character $\kappa : \mathbb{Z}_p^\times \to R^\times$ on the open locus of $\mathfrak{X}_R$ of ordinary reduction. The global sections of $\mathfrak{w}^\kappa$ are then the R-module of p-adic modular forms. Let us recall Serre's point of view on p-adic modular forms in Katz' reformulation, for the sake of an analogy we would like to make shortly:

Theorem 1.1.1 ([34], Theorem 4.5.1)**.** *For simplicity, let us assume $p > 2$.*

1. *Let $\chi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ be a p-adic weight. Let f be a p-adic modular form of weight χ and level $\Gamma(N)$ over $\mathbb{Z}_p$. Then there is a sequence of integers $0 \leq k_1 \leq k_2 \leq \dots$ that when regarded as p-adic weights $x \mapsto x^{k_m}$ satisfy $k_m \equiv \chi \bmod \varphi(p^m)$, and a sequence of classical modular forms f_m of weight k_m and level $\Gamma(N)$ defined over $\mathbb{Z}_p$ such that*
$$f_m \equiv f \bmod p^m \text{ in } q\text{-expansion.}$$
2. *Conversely, let $(k_m)_{m\in\mathbb{N}}$ be any sequence in $\mathbb{Z}$ and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of p-adic modular forms f_m of weight k_m and level $\Gamma(N)$ over $\mathbb{Z}_p$ such that $f_1 \not\equiv 0 \bmod p$ and*
$$f_{m+1} \equiv f_m \bmod p^m \text{ in } q\text{-expansion.}$$
Then the sequence $(k_m)_{n\in\mathbb{N}}$ converges in $\mathrm{End}(\mathbb{Z}_p^\times)$ to a weight $\chi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ and there is a unique p-adic modular form f of weight χ and level $\Gamma(N)$ over $\mathbb{Z}_p$ such that
$$f_m \equiv f \bmod p^m \text{ in } q\text{-expansion.}$$

A central result in the theory of p-adic modular forms is that classical Hecke eigenforms live in p-adic families of overconvergent p-adic eigenforms, as was first observed by Hida for ordinary modular forms [28], and later extended by Coleman [19], whose work with Mazur culminated in the construction of the eigencurve [18]. This is a rigid space which roughly can be described as follows: Let $W = \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\mathrm{rig}}_\eta$ be the rigid space over $\mathbb{Q}_p$ whose X-points for any affinoid rigid space X over $\mathbb{Q}_p$ parametrise p-adic weights $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$. Then the eigencurve is a rigid space $w : E \to W$ parametrising and interpolating overconvergent p-adic eigenforms of finite slope and level $\Gamma(N)$, relatively over the p-adic weight space W.

The geometry of the eigencurve encodes deep questions about congruences of eigenforms and Galois representations, and the eigencurve is therefore nowadays a central topic to the study of p-adic modular forms. Despite some recent progress, many of its geometric features are still mysterious. We refer to [33] for a brief survey.

We can picture the weight space W as being a disjoint union of open unit discs: Each point $\kappa \in W$ corresponding to a weight $\kappa : \mathbb{Z}_p^\times \to K^\times$ valued in some non-archimedean field extension K of $\mathbb{Q}_p$ can be assigned an absolute value $|\kappa| := \max_{a\in\mathbb{Z}_p^\times} |\kappa(a)-1|$ which we can use to define subdiscs and annuli in W of prescribed radii in each component of W.

Near the centre of the weight space discs, the geometry of the eigencurve is expected to be very complicated [12]. Near their boundary, on the other hand, the eigencurve is thought to behave in a much more "regular" fashion. Motivated by computational results, Coleman made the following conjecture:

Conjecture 1.1.2 (Coleman's Spectral Halo)**.** *There is $1 > r > 0$ such that over the boundary annuli $W^{>r} \subseteq W$ of weights κ with $1 > |\kappa| > r$, the eigencurve is a disjoint union of connected components $E^{>r} = \sqcup_{i=1}^{\infty} E_n^{>r}$ where each $w : E_n^{>r} \to W^{>r}$ is finite flat.*

The case of $p = 2$ and $N = 1$ of this conjecture was proved by Buzzard–Kilford [14], using explicit computations with Eisenstein series. They moreover found that there was an additional surprising pattern on the eigencurve: They could prove that over the open annulus $W^{r>1/8} \subseteq W$, the eigencurve decomposes into an infinite disjoint union of copies of the weight space, and that moreover for any given weight $\kappa \in W^{r>1/8}$, the 2-adic valuations of the eigenvalues of the U_2-operator, called slopes, on the space of overconvergent modular forms of weight κ lie in the arithmetic progression $0, s, 2s, 3s, \dots$, where $s = v_2(\kappa(5)-1)$ is a scaling factor which only depends on $|\kappa|$. The result of Buzzard–Kilford, complemented by computational evidence for other p and N, motivated the following conjecture:

Conjecture 1.1.3 (Buzzard–Kilford)**.** *There is a sequence $\lambda_1, \lambda_2, \dots$ such that for any $n \in \mathbb{N}$ and $z \in E_n^{>r}$, one has $v_p(a_p(z)) = \lambda_n v_p(w(z))$ where $a_p(z)$ is the U_p-eigenvalue. In particular, the slope of z only depends on $v_p(w(z))$. Moreover, the sequence λ is a finite union of arithmetic progressions.*

It has been speculated that this unexpected conjectural structure on the eigencurve might hint at additional symmetries in the spaces of p-adic modular forms near the boundary of weight space which are yet to be found (e.g. [3], question 1.1). For this and other reasons, the two conjectures have attracted considerable attention in recent years:

Roe [47], Kilford [40] and Kilford–McMurdy [41] could prove further instances of the conjecture in the cases of $p = 3, 5, 7$. In a recent breakthrough, Liu–Wan–Xiao could prove the analogous statement of both conjectures for quaternionic modular forms [42], thereby also proving most cases of the conjecture in the case of elliptic modular forms via the Jaquet–Langlands correspondence. In independent but related work, Bergdall–Pollack [4] proved that Coleman's conjecture implies the conjecture of Buzzard–Kilford.

For a different approach, Coleman suggested to study the eigencurve at the boundary of weight space using a compactification obtained by adding a "point in characteristic p" to each weight space discs which should be thought of as lying "on the boundary". The annulus $W^{>r}$ may then be regarded as a neighbourhood of the boundary point. Coleman conjectured that there should be a space of "overconvergent t-adic modular forms" for weights given by the continuous characters $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ obtained by reduction mod p of the universal weight. The t-adic eigenforms of finite slope should then form the boundary of families of p-adic eigenforms, in the sense that the eigencurve extends to the compactified weight space.

This strategy has recently been realised by Andreatta–Iovita–Pilloni [3]: In order to be able to work in a technical setting in which one can consider analytic spaces with points of different characteristics, the authors pass from the rigid space W to the analytic adic space $\mathcal{W} = \mathrm{Spa}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], \mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\mathrm{an}}$ in the sense of Huber. Additionally to the adification W^{ad} of W, the larger space $\mathcal{W}$ also contains points "in characteristic p", one for each weight space disc, corresponding to t-adic weights $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$. They then construct a line bundle $\omega^{\overline{\kappa}}$ on an overconvergent neighbourhood of the locus of ordinary reduction of the analytic modular curve over $\mathbb{F}_p((t))$. The global sections of $\omega^{\overline{\kappa}}$ are the space $M_{\overline{\kappa}}$ of t-adic overconvergent modular forms envisioned by Coleman. This $\mathbb{F}_p((t))$-Banach space has a natural Hecke action, and Andreatta–Iovita–Pilloni are able to show that, remarkably, this bundle can be glued to the sheaves of p-adic modular forms on the open discs to yield overconvergent sheaves of mixed characteristics families of modular forms over the adic weight space $\mathcal{W}$. The authors deduce that the eigencurve extends to an analytic adic space

$$\mathcal{E} \to \mathcal{W}$$

whose fibres over the boundary parametrise overconvergent t-adic eigenforms of finite slope.

The hope is now that a good understanding of the space of t-adic modular forms, and in particular of the fibre of $\mathcal{E}$ over the characteristic p point, would help resolve Coleman's Spectral Halo conjecture. More precisely, one hopes for a precise understanding of what happens on the eigencurve as we approach the characteristic p point on weight space.

We note that apart from the construction of Andreatta–Iovita–Pilloni, there have recently have been other, much more general constructions of extended eigenvarieties, by Johansson–Newton [31] and, independently, by Gulotta [26]. Instead of relying on sheaves of adic modular forms, both of these constructions use the theory of overconvergent cohomology, which they extend to the t-adic case.

1.2 Summary of the main result

The main goal of this work is to offer a new perspective on how the spaces of t-adic modular forms and p-adic modular forms near the boundary of weight space are related, building on the work of Andreatta–Iovita–Pilloni: Using methods from Scholze's theory of perfectoid spaces [48], we describe the Hecke-module of Andreatta–Iovita–Pilloni's overconvergent t-adic modular forms purely and explicitly in terms of overconvergent p-adic modular forms.

The language of perfectoid spaces has entered the world of p-adic modular forms already, for example in the description due to Chojecki–Hansen–Johansson [17] of p-adic modular forms as functions on open subspaces of the perfectoid infinite level modular curve $\mathcal{X}_{\Gamma(p^\infty)}$. We show that this is not just a technical coincidence, but that there are "perfectoid phenomena" occuring in the theory of modular forms near the boundary of weight space.

To explain what we mean by that, let K be a perfectoid field extension of $\mathbb{Q}_p$ and let R be a perfectoid K-algebra. A basic construction in the theory of perfectoid algebras introduced by Scholze [48] is the tilting functor, which associates to R a perfectoid algebra $R^\flat$ over the perfectoid field $K^\flat$ of characteristic p by considering sequences $(x_n)_{n\in\mathbb{N}}$ of power-bounded elements $x_n \in R^\circ$ such that $x_{n+1}^p \equiv x_n \bmod p$. More precisely, $R^\flat$ is defined by

$$R^{\flat\circ} := \varprojlim_{x\mapsto x^p} R^\circ/p, \qquad R^\flat := R^{\flat\circ}[1/t]$$

where we let t be a pseudo-uniformiser of $K^\flat$.

The main result of this work is that a similar construction works for modular forms, when we replace perfectoid K-algebras by p-adic modular forms and perfectoid $K^\flat$-algebras by t-adic modular forms. Roughly, the space of t-adic modular forms at the boundary can be described as compatible sequences f_n of p-adic modular forms of weights converging to the boundary satisfying $f_{n+1}^p \equiv f_n \bmod p$. We note, however, that it is clear from looking at q-expansions that this cannot quite work exactly as we just stated: The algebra $K[[q]]$ is not perfectoid, because it lacks p-th roots of q mod p. From this perspective of q-expansions, there are two naive ways to fix this: The first is to allow q-expansions in the perfectoid algebra $K[[q^{1/p^\infty}]]$, the second, more refined one, is to work with the subalgebras of the former of the form $K[[q^{1/p^n}]]$, where we let n grow as we go up the inverse system: Mod p, we then have Frobenius maps $\mathcal{O}_K/p[[q^{1/p^{n+1}}]] \to \mathcal{O}_K/p[[q^{1/p^n}]]$ which we can use to make sense of p-th roots of modular forms. It turns out that from a more geometric perspective, both of these constructions can be given a sense, and one can indeed define arithmetically interesting spaces of p-adic modular forms whose q-expansions are of the forms just described.

To make this precise, let now K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. We then define for any p-adic character $\kappa \in \mathcal{W}$ valued in K an invertible $\mathcal{O}^+$-module $\omega^{\kappa,+,\mathrm{perf}}$ on the base change to K of the perfectoid modular curve $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ from [50], whose sections yield a large $\mathcal{O}_K$-module $M_\kappa^{+,\mathrm{perf}}(\epsilon)$ of "integral overconvergent perfectoid modular forms". This contains the usual space of integral overconvergent modular forms as a subspace $M_\kappa^+(\epsilon) \subseteq M_\kappa^{+,\mathrm{perf}}(\epsilon)$, and can indeed be regarded as the "perfection" of $M_\kappa^+(\epsilon)$ in a precise sense. For any $n \in \mathbb{N}$, it also contains the subspace $M_{\kappa,\Gamma_0(p^n)}(\epsilon)^+$ of modular forms of level $\Gamma_0(p^n)$, which we define to be the sections of a line bundle $\omega^{\kappa,+}$ on $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$. In terms of the point of view of q-expansions, modular forms in $M^+_{\kappa,\Gamma_0(p^n)}(\epsilon)$ have expansions of the form $\mathcal{O}_K[[q^{1/p^n}]]$, while elements of the space $M_\kappa^{+,\mathrm{perf}}(\epsilon)$ have "perfectoid" expansions in $\mathcal{O}_K[[q^{1/p^\infty}]]$.

Slightly generalising the construction from [3], one can make analogous definitions in characteristic p: Let $\mathcal{X}'$ be the compactified modular curve over $K^\flat$, considered as an analytic adic space. Let $\mathcal{X}'(\epsilon)$ be the ϵ-neighbourhood of the ordinary locus. For any continuous character of the form $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$, we define an invertible $\mathcal{O}^+$-module $\omega^{\kappa^\flat,+}$ on $\mathcal{X}'(\epsilon)$ whose sections give the space $M^+_{\kappa^\flat}(\epsilon)$ of t-adic overconvergent modular forms of weight $\kappa^\flat$. By pullback one obtains a bundle $\omega^{\kappa^\flat,+,\mathrm{perf}}$ on the perfected space $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$ over $K^\flat$, whose sections are the $\mathcal{O}_{K^\flat}$-module $M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)$ of overconvergent integral perfectoid modular forms of weight $\kappa^\flat$. This contains the space of t-adic forms $M^+_{\kappa^\flat}(\epsilon) \subseteq M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)$ as a subspace.

Our main result is now the following "tilting isomorphism of modular forms". We refer to Theorem 12.1.1 and Theorem 12.2.1 below for slightly more precise statements.

Theorem 1.2.1. *Let $(\kappa_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of p-adic weights $\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ such that $|T_{\kappa_0}| \geq |p|$ and $|T_{\kappa_1}| > |p|^{1/(p-1)}$ and $\kappa_{n+1}^p \equiv \kappa_n \bmod p$. Via the identification $\mathcal{O}_{K^\flat}^\times = \varprojlim_{x\mapsto x^p}(\mathcal{O}_K/p)^\times$, this defines a weight over $K^\flat$ that we denote by $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$. Then for small enough radii of overconvergence $0 \leq \epsilon$, we have:*

1. *There is a natural almost isomorphism of $\mathcal{O}_{K^\flat} = \varprojlim_F \mathcal{O}_K/p$-modules*
$$M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon) \stackrel{a}{=} \varprojlim_{f\mapsto f^p} M^{+,\mathrm{perf}}_{\kappa_n}(\epsilon)/p.$$
2. *There is $0 \leq l < 1$ with $l = O(\epsilon)$ such that this restricts to an almost isomorphism*
$$M^+_{\kappa^\flat}(\epsilon) \stackrel{a}{=} \varprojlim_{f\mapsto f^p} M^+_{\kappa_n,\Gamma_0(p^n)}(\epsilon)/p^{1-l} = \varprojlim_{f\mapsto f^{(p)}} M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}$$
that is equivariant for the action of the integral Hecke algebra.

Remark 1.2.2. The conditions on the κ_n imply that $|T_{\kappa_{n+1}}|^p = |T_{\kappa_n}|$. In particular, the sequence of κ_n automatically converges to the boundary of weight space, see Remark 12.2.2.

We think of Theorem 1.2.1 as a way to describe how modular forms near the boundary of weight space "converge into characteristic p", much in the same way that p-adic modular forms can be written as p-adic limits of classical modular forms. Let us make this precise in the case of $\epsilon = 0$, which is the case of convergent p-adic modular forms (rather than overconvergent ones). This case is much easier to understand for a variety of technical reasons, including the fact that we can often argue with q-expansions, as explained in §5.3. In this situation, we understand the above tilting result to say that there is an analogue of Katz' Theorem 1.1.1 for the boundary of weight space, using "t-adic convergence" instead of

p-adic convergence, where t denotes a pseudo-uniformiser of $\mathcal{O}_{K^\flat}$ such that $\mathcal{O}_K^\flat/t = \mathcal{O}_K/p$. By this we mean the following: With Serre's convergence of modular forms in mind, we may regard a sequence of elements $(x_m)_{m\in\mathbb{N}}$ with $x_m \in \mathcal{O}_K$ as p-adically convergent if the reductions define an element in $\mathcal{O}_K = \varprojlim_m \mathcal{O}_K/p^m$. In this light, one could say that a sequence of elements in $\mathcal{O}_K$ is t-adically convergent if the reductions mod p are instead contained in $\mathcal{O}_{K^\flat} = \varprojlim_m \mathcal{O}_{K^\flat}/t^m = \varprojlim_F \mathcal{O}_K/p$. In other words, this means that

$$x_{m+1}^p \equiv x_m \bmod p$$

for all $m \in \mathbb{N}$. We shall also call $(x_m)_{m\in\mathbb{N}}$ a Frobenius-compatible sequence.

From this perspective, we can reinterpret the case of $\epsilon = 0$ of Theorem 1.2.1 as follows:

Definition 1.2.3. For any $f = \sum a_n q^n$ in $\mathcal{O}_K[[q]]$, we define $f^{(p)} := \sum a_n^p q^n$.

Corollary 1.2.4 (t-adic modular forms from the perspective of Serre convergence)**.**

1. *Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be a weight and let $f^\flat$ be a t-adic modular form of weight $\kappa^\flat$, i.e. an element of $M_{\kappa^\flat}^+(0)$. Then there is a sequence $(\kappa_n)_{n\in\mathbb{N}}$ of weights $\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ with $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ for all n, converging to the boundary of weight space, and a sequence of p-adic modular forms $f_n \in M_{\kappa_n}^+(0)$ with q-expansions satisfying*

$$f_{n+1}^{(p)} \equiv f_n \bmod p,$$

for all n, such that the f_n converge to $f^\flat$ with respect to $\mathcal{O}_{K^\flat}[[q]] = \varprojlim_{f\mapsto f^{(p)}} \mathcal{O}_K/p[[q]]$.

2. *Conversely, let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence of weights $\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ converging to the boundary of weight space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of p-adic modular forms $f_n \in M_{\kappa_n}^+(0)$ over $\mathcal{O}_K$ of weights κ_n with q-expansions satisfying*

$$f_{n+1}^{(p)} \equiv f_n \bmod p$$

for all n. Assume that $|f_1| = 1$ (meaning that at least one q-expansion coefficient of f_1 is in $\mathcal{O}_K^\times$). Then $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ for all n, and there is a unique t-adic modular form $f^\flat$ defined over $\mathcal{O}_{K^\flat}$ of character $\kappa^\flat$ associated to $(\kappa_n)_{n\in\mathbb{N}}$ such that via the isomorphism $\mathcal{O}_{K^\flat}[[q]] = \varprojlim_{f\mapsto f^{(p)}} \mathcal{O}_K/p[[q]]$, the f_n converge to $f^\flat$ in q-expansions .

Remark 1.2.5. The Theorem also gives a way to distinguish the spaces of t-adic modular forms of different t-adic weights: All weights of the form $\kappa : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ correspond to just a finite number of boundary points of the adic weight space $\mathcal{W}$, determined by the torsion part $\kappa : \mu_{p-1}(\mathbb{Z}_p) \to \mathbb{F}_p^\times$ of the weight. As explained in detail in §10, the reason for this is that any two weights with fixed torsion part are related by an automorphism of $\mathbb{F}_p((t))$, and in particular give rise to isomorphic Hecke-modules of t-adic modular forms.

In light of the tilting equivalence, however, the specific t-adic weight can be interpreted as remembering the choice of Frobenius-compatible sequence converging to the boundary.

Even though at this point in time we do not have any immediate applications to the conjectures of Coleman and Buzzard–Kilford, we hope that ultimately this tilting equivalence of modular forms helps understand conceptually why Coleman's Spectral Halo appears.

Finally in this section, we would like to mention that we believe that analogous tilting equivalences exist also for p-adic automorphic forms for other reductive groups that have Shimura varieties with nonempty ordinary locus, and that our strategies should generalise. For example, we do not actually make crucial use (i.e. other than for expository purposes) of q-expansion principles or other techniques that are restricted to the modular curve case. In particular, we believe that perfectoid modular forms should exist in greater generality, and that our main tools to study them as described in the next section should generalise.

As a concrete example, we believe that it should be possible to generalise these results in the case of Hilbert modular forms, based on the objects defined in [2] and [10]. Since the boundary of the adic eigenvariety in this case is much larger, and investigations so far suggest that there is a rich structure to discover [9][59], it would be interesting to see what one might learn in this context from tilting isomorphisms.

1.3 Method of proof, and further results

We shall now sketch how the tilting isomorphism arises, and give an outline of the main ingredients we develop for the proof.

1.3.1 Perfectoid modular curves and perfectoid modular forms

As a first step, we need to adapt the constructions of Andreatta–Iovita–Pilloni of bundles of p-adic and t-adic modular forms to the context of a perfectoid base field K.

In case that K is an extension of $\mathbb{Q}_p^{\mathrm{cyc}}$, this basically amounts to a slight modification of the definition by Chojecki–Hansen–Johansson [17], which we describe in detail in §2. The main way in which we deviate from their setup is that we choose to work with the anticanonical locus $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ of the perfectoid modular curve at infinite level, while [17] works with the canonical one. Working with the anticanonical locus is more convenient for us since it is easier to compare to modular curves over $K^\flat$. It also has the advantage that it is easier to discuss q-expansions since the cusps are easier to describe in the anticanonical tower. Second, we explain how the "formes modulaires perfectisées" of [3], §6 appear in this context: This is what we call perfectoid modular forms, motivated by the tilting isomorphism. In our setting they can be described as functions on the pro-étale $\mathbb{Z}_p^\times$-torsor

$$\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a,$$

whereas p-adic modular forms are defined using instead for $n < \infty$ the pro-étale morphism

$$\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a,$$

which is a pro-étale $\Gamma_0(p^n)$-torsor away from the cusps. More precisely, we construct for any weight $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ sheaves of modular forms $\omega^\kappa_{\Gamma_0(p^n)}$ on $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ for all $n \in \mathbb{N}$, as well as a sheaf of perfectoid modular forms $\omega^{\kappa,\mathrm{perf}}$ on the affinoid perfectoid space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$. Using the integral structure sheaves of these adic spaces, one obtains natural integral versions $\omega^{\kappa,+}_{\Gamma_0(p^n)}$ and $\omega^{\kappa,+,\mathrm{perf}}$ of these sheaves, and it turns out that these are invertible $\mathcal{O}^+$-modules.

Based on [27], we also discuss q-expansions of p-adic and perfectoid modular forms, including various q-expansion principles, which in analogy with Katz' q-expansion principle [34] allow one to detect properties of modular forms like vanishing, integrality and the level at p from the q-expansion. However, we only introduce these to be able to give some illustrating remarks, and do not use them as a technical ingredient in the tilting isomorphism.

In section 3, we adapt in a similar fashion the definition of modular forms and what we call perfectoid modular forms over a perfectoid field $K^\flat$ of characteristic p. Let $\mathcal{X}'(\epsilon)$ be the ϵ-neighborhood of the ordinary locus in the rigid analytic modular curve of $K^\flat$ – here we follow a convention from [50] and decorate spaces over $K^\flat$ with an additional $'$ to distinguish them from their counterparts in characteristic 0. For the definition of t-adic perfectoid modular forms, one uses the perfectoid Igusa tower, which in this setting has an incarnation as a $\mathbb{Z}_p^\times$-torsor of perfectoid spaces

$$\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}.$$

For technical reasons, the definition of true t-adic modular forms is slightly more involved: One would like to replace the above map by an analytic infinite Igusa space that is a $\mathbb{Z}_p^\times$-torsor over $\mathcal{X}'(\epsilon)$. Since for $\epsilon > 0$ it is not clear that this is possible in the category of adic spaces, one instead resorts to the category of formal schemes where one has a morphism

$$\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) \to \mathfrak{X}'(\epsilon)$$

that is to be thought of as a formal model of an analytic infinite Igusa space over $\mathcal{X}'(\epsilon)$. We thus get for any t-adic weight $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ a sheaf of t-adic modular forms $\mathfrak{w}^{\kappa^\flat}$ on $\mathfrak{X}'(\epsilon)$ as well as a sheaf of t-adic perfectoid modular forms $\mathfrak{w}^{\kappa^\flat,\mathrm{perf}}$ on $\mathfrak{X}'(\epsilon)^{\mathrm{perf}}$. Their analytifications over $\mathcal{X}'(\epsilon)$ give rise to sheaves $\omega^{\kappa^\flat,+}$ and $\omega^{\kappa^\flat}$ like in the p-adic setting. There is also analogous theory of cusps and q-expansions.

Before we compare sheaves over K to those over $K^\flat$, we discuss in section 4 in each characteristic p and 0 separately how the different sheaves in each case are related: We first compare different notions of what it means for our p-adic and t-adic perfectoid modular forms to be integral. For this, we also define formal models of the p-adic modular sheaves. We then show that both in the p-adic and t-adic case, one can use the formal model $\mathfrak{w}^\kappa$ and the analytic integral sheaf $\omega^{\kappa,+}$ interchangeably, since they can be reconstructed from one another. This boils down to saying that for example the formal model $\mathfrak{X}(\epsilon)$ is integrally closed in its generic fibre. We note that this is a non-Noetherian formal scheme, so the usual algebraic criteria for normality do not apply. Instead, we give what could be described as a sort of "perfectoid criterion" in this setting for being integrally closed in the generic fibre.

As the second goal in section 4, we adapt from [3] an important technical tool that essentially allows one to switch back and forth between modular forms and perfectoid modular forms: There are natural "trace" maps, namely continuous $\mathcal{O}_K$-linear sections

$$M_\kappa^{+,\mathrm{perf}}(\epsilon) \to p^{-\delta} M^+(\epsilon)$$

for some explicit $\delta = \delta(\epsilon) \geq 0$ of the natural inclusion $M^+(\epsilon) \to M^{+,\mathrm{perf}}(\epsilon)$. These have their origins in Scholze's normalised Tate traces, [50] III.2.4, which played a technical role in the construction of the modular curve at infinite level.

Finally in §4, we discuss the change of base field K for integral modular forms, since we will need this as a reduction step in the proof of the tilting isomorphism.

1.3.2 The tilting isomorphism of modular curves

In chapter 5, we introduce the first weak version of our tilting isomorphism of modular forms. The starting point for this is Corollary III.2.19 in [50] which in our notation says that there is a canonical tilting isomorphism

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a^\flat \xrightarrow{\sim} \mathcal{X}'(\epsilon)^{\mathrm{perf}}.$$

Complementing a result of [3] for higher dimensional Siegel varieties in the case of modular curves, we show in [27] that this tilting isomorphism extends to a canonical isomorphism

$$\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a^\flat \xrightarrow{\sim} \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}.$$

It is this isomorphism, together with the affinoid perfectoidness of these two spaces, which proves the first version of a tilting equivalence for perfectoid modular forms: Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be any weight. Then for $\epsilon \geq 0$ small enough there is a natural isomorphism of $\mathcal{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p$-modules

$$M_\kappa^{\mathrm{perf},+}(\epsilon) = \varprojlim_{x\mapsto x^p} M_{\kappa_n^\sharp}^{\mathrm{perf},+}(\epsilon)/p.$$

From this we can deduce a version of our desired general tilting isomorphism in the case of $\epsilon = 0$: In other words, we can easily trade in the overconvergence in the above isomorphism for flexibility in the weights and get a version for true modular forms on top, namely the convergent tilting isomorphism. This can be done using q-expansion principle and Eisenstein series, but we will later prove a stronger version without these methods. Nevertheless, this is already enough to prove the Serre-style convergence result, Theorem 1.2.4 above.

The goal of sections §6-11 could now be described as being an extensive approach to showing that this tilting isomorphism overconverges. We do this by systematically developing various related perfectoid aspects of the theory of families of modular forms, which might be of independent interest beyond their applications to the proof of the tilting isomorphism. Our analysis can be divided into constructing two separate tools:

1.3.3 First main ingredient: the Almost Acyclicity Theorem

To motivate the first ingredient, let us first mention the following easy consequence of the tilting isomorphism of modular curves (see Lemma 11.2.1 for a more precise statement)

Lemma 1.3.1. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ and $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be two characters such that $\kappa \equiv \kappa^\flat \bmod p$. Then we have a natural identification of sheaves on $|\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a| = |\mathcal{X}'(\epsilon)^{\mathrm{perf}}|$*

$$\omega^{\kappa^\flat,+,\mathrm{perf}}/t \overset{a}{=} \omega^{\kappa,+,\mathrm{perf}}/p.$$

It follows from this that for any sequence of p-adic weights $\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ satisfying $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ converging to a weight $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$, we have the following "sheaf version" of the perfectoid overconvergent tilting isomorphism (see Theorem 12.1.1.2 for a precise statement): There is a natural almost isomorphism

$$\omega^{\kappa^\flat,+,\mathrm{perf}} \overset{a}{=} \varprojlim_{x\mapsto x^p} \omega^{\kappa_n,+,\mathrm{perf}}/p$$

of $\mathcal{O}^{+,a}_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}$-modules on $|\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a| = |\mathcal{X}'(\epsilon)^{\mathrm{perf}}|$. In order to deduce the tilting isomorphism of perfectoid modular forms, we would like to be able to commute the functors of global section and reduction mod p on the right hand side. This is possible by the following almost acyclicity result, our first main ingredient:

Theorem 1.3.2 (Theorem 11.0.2)**.** *Let K be a perfectoid field over either $\mathbb{Q}_p^{\mathrm{cyc}}$ or $\mathbb{Q}_p^{\mathrm{cyc}\flat}$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight. Then for $\epsilon > 0$ small enough, the sheaf $\omega^{\kappa,+,\mathrm{perf}}$ on $|\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a| = |\mathcal{X}'(\epsilon)^{\mathrm{perf}}|$ is almost acyclic, i.e. $H^q(\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a, \omega^{\kappa,+,\mathrm{perf}}) \overset{a}{=} 0$ for all $q > 0$.*

The natural starting point for our proof is to follow the strategy of the proof of almost acyclicity of the structure sheaf $\mathcal{O}_Y^+$ of an affinoid perfectoid space Y from [48]: The idea of the latter is to start with the case of characteristic p, where in the case that Y can be written as the perfection $Y = Y_0^{\mathrm{perf}}$ of some reduced affinoid rigid space Y_0, one can exhibit the Čech complex of $\mathcal{O}_Y^+$ for some open cover of Y as the completed ind-perfection of the Čech complex of Y_0. Since the cohomology of the latter has bounded t-torsion by a topological algebra lemma, this implies that the cohomology of the former is almost annihilated.

To adapt this strategy to the sheaves of perfectoid modular forms, we instead consider over $K^\flat$ the family of sheaves $(\mathfrak{w}^{\kappa^{p^n}})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$ and show that we can regard the formal model $\mathfrak{w}^{\kappa,\mathrm{perf}}$ of $\omega^{\kappa,+,\mathrm{perf}}$ as being the t-adically completed "ind-perfection"

$$\mathfrak{w}^{\kappa,\mathrm{perf}} = (\varinjlim_{x\mapsto x^p} \mathfrak{w}^{\kappa^{p^n}})^\wedge.$$

This reduces us to proving that torsion in the Čech cohomology of the $\mathfrak{w}^{\kappa^{p^n}}$ is uniformly bounded, i.e. it is annihilated by some t^N where N is independent of n. In the case of the structure sheaf $\mathcal{O}_Y^+$ considered in [48], an instance of which appears here as the case of $\kappa = 1$, this uniformity is immediate since the sheaves $(\mathcal{O}_Y^+)^{\otimes p^n} = \mathcal{O}_Y^+$ are all the same.

To see that torsion in the Čech complex for this family can be uniformly bounded, we need more arithmetic input: To this end, we develop in Chapter 9 a theory of analytic families of t-adic overconvergent modular forms over the t-adic weight space

$$\mathcal{W}' = \mathrm{Spa}(\mathbb{F}_p[[t]][[\mathbb{Z}_p^\times]], \mathbb{F}_p[[t]][[\mathbb{Z}_p^\times]])^{\mathrm{an}} = \sqcup_{\mathrm{End}(\mathbb{F}_p^\times)} \mathrm{Spa}(\mathbb{F}_p[[t]][[T]])^{\mathrm{an}}.$$

Here the disjoint union is labelled over the finitely many characters of the form $\mathbb{F}_p^\times \to \mathbb{F}_p^\times$. We think of $\mathcal{W}'$ as an equicharacteristic analogue of the p-adic weight space, with $\mathbb{Z}_p$ replaced by $\mathbb{F}_p[[t]]$. Based on the definition of t-adic modular forms, we define a sheaf $\mathfrak{w}'$ of integral families of overconvergent modular forms over $\mathcal{W}'$. We also explain why all non-trivial characters $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ with some fixed restriction $\kappa : \mathbb{F}_p^\times \to \mathcal{O}_{K^\flat}^\times$ correspond to just one boundary point of the p-adic weight space: The answer is that the entire sheaf $\mathfrak{w}'$ of modular forms over $\mathcal{W}'$ can be obtained by pullback from the sheaf of modular forms of just one single "canonical" weight by swapping the roles of T as a weight space parameter and t as a uniformiser of $\mathbb{F}_p[[t]]$. In other words, we use automorphisms of $\mathbb{F}_p[[t]][[T]]$ that have no analogue for $\mathbb{Z}_p[[T]]$. These automorphisms make it possible to "collapse" the t-adic weight discs to finitely many points without loosing information, which is what remains of it in $\mathcal{W}$.

Returning to the proof of the Acyclicity Theorem, the idea is now that after slightly changing the condition on the Hasse invariant to a "uniform" condition independent of the weight variable, thereby passing from the sheaf $\mathfrak{w}'$ to a "uniform" sheaf $\mathfrak{w}'^{\mathrm{u}}$, we can recover all bundles $\mathfrak{w}^{\kappa^{p^n}}$ from the bundle $\mathfrak{w}'^{\mathrm{u}}$ by specialisation. Since one can bound the torsion in the Čech cohomology of $\mathfrak{w}'^{\mathrm{u}}$, this allows one to bound the torsion for all $\mathfrak{w}^{\kappa^{p^n}}$ uniformly.

This will complete the proof of the Acyclicity Theorem in the case of characteristic p. The case of characteristic 0 follows by a homological algebra argument. At this point, we can deduce the perfectoid version of the overconvergent tilting isomorphism, Theorem 1.2.1.1.

1.3.4 The second main ingredient: Canonical lifts at the boundary

Having completed the proof of the tilting isomorphism of perfectoid modular forms, we are left to deduce Theorem 1.2.1.2 from the first part. For this it does not suffice to just "apply traces": firstly, because these do not quite preserve the integral subspaces, second because these will not tell us anything about Hecke equivariance. What the traces do allow us to do (via Lemma 12.2.6) is to reduce to constructing an $\mathcal{O}_{K^\flat}$-linear Hecke-equivariant map

$$M^+_{\kappa^\flat}(\epsilon) \to M^+_\kappa(\epsilon)/p^{1-l}.$$

Since the p-torsion in the higher cohomology of $\omega^{\kappa,+}$ is probably large, this basically amounts to associating to $f^\flat \in M^+_{\kappa^\flat}(\epsilon)$ a modular form in $M^+_\kappa(\epsilon)$. In other words, we would like to construct a map of sets

$$M^+_{\kappa^\flat}(\epsilon) \to M^+_\kappa(\epsilon) \tag{1}$$

from which we obtain the above map mod p by composing with reduction mod p.

In the analogy between perfectoid algebras and modular forms, this map should roughly correspond to the sharp map $\sharp : R^{\flat\circ} \to R^\circ$. In order to follow this intuition, we recall that the $\sharp$-map could be described as being the composition

$$\begin{tikzcd}
 & W(R^{\flat\circ}) \arrow[dl] \arrow[dr, "\theta"] & \\
R^{\flat\circ} \arrow[ur, bend left, "{[-]}"] \arrow[rr, dotted, "\sharp"] \arrow[dr] & & R^\circ \arrow[dl] \\
 & R^{\flat\circ}/t = R^\circ/p. &
\end{tikzcd}$$

of the canonical multiplicative lift $[-] : R^{\flat\circ} \to W(R^{\flat\circ})$ with the natural map θ.

Our second main goal could be described as making sense of this diagram for modular forms: Recall that perfectoid t-adic modular forms live over the field $\mathbb{Q}_p^{\mathrm{cyc}\flat} = \mathbb{F}_p((t^{1/p^\infty}))$. The first key observation is that we can identify $W(\mathbb{F}_p[[t^{1/p^\infty}]])$ with the topological $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[[(1+T)^{1/p^\infty}]]$ whose associated (p,T)-adic formal scheme can be interpreted as a component of the cover $\mathfrak{W}_\infty \to \mathfrak{W} = \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$ parametrising sequences of weights $(\kappa_n)_{n\in\mathbb{N}}$ with $\kappa_{n+1}^p = \kappa_n$, like in the tilting isomorphism of modular forms.

The correct replacement for $W(R^{\flat\circ})$ in the setting of modular forms should therefore be a sheaf of integral families of perfectoid modular forms over this perfected weight space. We construct this bundle, which for reasons to become clear later we will denote by $\omega^+_{\infty,\infty,l}$, step by step in sections §6-§9. Morally, this sheaf is just the base change to $\mathfrak{W}_\infty$ of the sheaf that Andreatta–Iovita–Pilloni construct in [3] §6 as an intermediate step of the construction of the sheaf of mixed characteristic families of true modular forms. However, it requires some algebraic work to show that this base change has the right technical properties. There will also be a sheaf of true modular forms $\omega^+_{r,\infty,l}$ over the perfected weight space.

The analogue of the reduction map $W(R^{\flat\circ}) \to R^{\flat\circ}$ in this setting is then the specialisation map $\mathcal{O}(\omega^+_{\infty,\infty,l}) \to \mathcal{O}(\omega^{\kappa^\flat,+,\mathrm{perf}}) = M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)$ that sends a family over some open weight space annulus $\mathcal{W}_l \subseteq \mathcal{W}$ to its value at the boundary. Similarly, the analogue of θ is the specialisation at κ, or more precisely at a sequence of weights $(\kappa_n)_{n\in\mathbb{N}}$ with $\kappa_1 = \kappa$.

Following a suggestion of Pilloni, our second main ingredient is now to show that there is also an analogue of $[-]$ in this setting, namely a "canonical" lift of t-adic modular forms at the boundary: More precisely, we construct a canonical multiplicative section

$$[-] : M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon) \to \mathcal{O}(\omega^+_{\infty,\infty,l})$$

of the specialisation map. In particular, we can now by analogy define a $\sharp$-map by taking a modular form at the boundary, lifting it canonically to a family, then specialising at κ.

For the missing link in the proof of the tilting isomorphism we would instead like such a map for true t-adic modular forms. In a second step we therefore use traces to also construct a set-theoretic section

$$[-]_r : M^+_{\kappa^\flat} \to \mathcal{O}(\omega^+_{r,\infty,l})$$

of the specialisation map that extends a modular forms to a family of true modular forms of some uniform radius of overconvergence parametrised by the index variable r. Specialising $[f]_r$ at κ then gives the desired map (1) which we shall denote by $\natural : M^+_{\kappa^\flat}(\epsilon) \to M^+_\kappa(\epsilon)$. Since there is a good theory of Hecke operators for families of modular forms, and specialisation is clearly Hecke-equivariant, the existence of this map will complete the proof of Theorem 1.2.1.

In order to construct the lift $[-]$, we need quite a bit of technical input, which we shall now briefly describe: In chapter 6 we adapt the constructions of families of modular forms from [3] to construct formal models of infinite level modular curves over the perfected weight space. In doing so, we have to be careful because the formal schemes involved in this are not Noetherian anymore, so that we need to take extra care in several places: We do so by approximating the perfected weight space by spaces representing p^k-th roots of κ, and then let $k \to \infty$. The main object of study of this section are therefore the Igusa curves

$$\mathfrak{IG}_{n,r,k,l} \to \mathfrak{X}_{r,k,l}$$

where l is a weight space parameter, k indicates that we are working over the p^k-th root weight space, r is a radius of overconvergence, and p^n is the rank of the Igusa level structure.

In section 7, we complete several related technical tasks: The first is to prove several invariance lemmas for the Galois action on the Igusa schemes that we need in order to define modular sheaves. Second, we show how $\mathfrak{IG}_{\infty,\infty,\infty,l}$ can be related to the (p,T)-adic Witt-vector lift of $\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$. This will later enable us to construct the canonical lift.

In order to actually be able to define modular sheaves over the perfected weight space, it does not suffice to consider these formal schemes, and we need to work in the analytic topology. In order to make sure that this is well-defined, we show in section 8 that the adic generic fibre $\mathcal{X}_{r,\infty,l}$ of $\mathfrak{X}_{r,\infty,l}$ is sousperfectoid, and thus sheafy. In proving this, we also construct the various trace maps for modular forms (closely following [3] and [50]) that we have already mentioned several times.

Putting everything together, we can then in section 9 construct the desired modular sheaves $\omega^+_{r,\infty,l}$ on $\mathcal{X}_{r,\infty,l}$ using Igusa towers over the perfected weight space. The characteristic feature of this setting is that this modular sheaf has p^k-th roots with respect to the operation of tensor powers, which puts us into a perfectoid context. At this stage, we can already construct the perfectoid version of the canonical lift, namely for $\omega^+_{\infty,\infty,l}$

In order to get the second version for the sheaves $\omega^+_{r,\infty,l}$ that we need to construct the map (1), we would like to use trace maps. As mentioned above, these do not in general preserve integrality. The crucial observation is now that if we know the specialisation at the boundary is a finite level modular form, one can always ensure that the trace is integral at the expense of slightly decreasing the annulus of weight space we work on:

Proposition 1.3.3 (Proposition 9.3.2). *Let $f \in \mathcal{O}(\omega^+_{\infty,\infty,l})$ be such that its image at the boundary, i.e. in $\mathcal{O}(\omega^+_{\infty,\infty,\infty})$, is already contained in $\mathcal{O}(\omega^+_{r,\infty,\infty})$. Then for any $l' \in 1/p^m\mathbb{Z}_{\geq 0}$ with $l' \geq l+3/p^{r-2}(p-1)$, the restriction of $\mathrm{tr}_r(f)$ to $\mathcal{X}_{r,\infty,l'} \hookrightarrow \mathcal{X}_{r,\infty,l}$ is integral, i.e. it is already contained in $\mathcal{O}(\omega^+_{r,\infty,l'}) \subseteq \mathcal{O}(\omega_{r,\infty,l'})$.*

This allows us to finish the construction of the canonical lift, and thus of the map (1).

Remark 1.3.4. As already mentioned in regards to the tilting equivalence, the case of convergent t-adic modular forms, i.e. the case of $\epsilon = 0$, is much easier to understand, using q-expansion principles. For two other examples, we can use that $\mathfrak{X}'(0)$ is affine to substantially simplify the proof of the Acyclicity Theorem for $\epsilon = 0$ by reducing to the Noetherian case where the line bundles $\mathfrak{w}^{p^n}$ are acyclic on the ordinary locus by Cartan's Theorem B. Second, over the ordinary locus we have a full infinite level Igusa tower even in characteristic 0, so that already the construction of families of convergent p-adic modular forms is much simpler, and the construction of the canonical lift becomes straight-forward.

In section 12, we then put everything together and prove the overconvergent tilting equivalence, making precise the strategy we have sketched in this introduction.

In section 13, we discuss the relation of our work to the geometry of the eigencurve, and formulate a conjecture that links the tilting isomorphism to the Spectral Halo Conjecture.

There is an Appendix A in which we collect several results in topological algebra that we need throughout, e.g. permanence properties of Huber pairs in completed direct limits.

Finally, Appendix B summarises our results in a table of analogies, illustrating how closely the spaces $M^{+,\mathrm{perf}}_\kappa(\epsilon)$ and $M^+_\kappa(\epsilon)$ resemble perfectoid algebras.

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Notation and conventions

For any non-archimedean field K, we denote by $\mathcal{O}_K$ the ring of integers.

We use adic spaces in the sense of Huber [30], i.e. in the language of [52], we will only consider sheafy adic spaces. We usually identify rigid spaces with their adic analytification.

On the other hand, we do want to distinguish formal schemes $\mathfrak{X}$ over $\mathcal{O}_K$ from the adic space $\mathfrak{X}^{\mathrm{ad}}$ attached to them. This is for two main reasons: Firstly, our formal schemes are often non-Noetherian, so that it is not clear if $\mathfrak{X}^{\mathrm{ad}}$ is sheafy. Second, we often work in the formal scheme topology rather than in the much finer topology of the associated adic space in the sense of Scholze–Weinstein. We denote by $\mathfrak{X}^{\mathrm{ad}}_\eta$ the adic generic fibre over $\mathrm{Spa}(K, \mathcal{O}_K)$.

For the most part of this work, and in particular for the context of the main theorem, we use perfectoid fields and perfectoid algebras in the sense of [48]. Only from §8 onwards we will also want to use of the more general notion of perfectoid algebras defined in [53] which does not require a perfectoid base field.

We let $\mathbb{Q}_p^{\mathrm{cyc}}$ be the field obtained by adjoining to $\mathbb{Q}_p$ all p-th roots of unity and completing.

In the context of adic spaces, we use tilde-limits in the sense of [52], Definition 2.4.1. If K is any perfectoid field, and $G = \varprojlim G_n$ is any profinite group, we let $\underline{G}$ be the unique perfectoid tilde-limit $\underline{G} \sim \varprojlim_n \underline{G_n}$ as defined in [53], §9.3. Explicitly, we let $\underline{G} = \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(G, K), \mathrm{Map}_{\mathrm{cts}}(G, \mathcal{O}_K))$. This is a group object in adic spaces over $(K, \mathcal{O}_K)$.

When working over non-discrete valuation rings, and in particular over $\mathcal{O}_K$ for K a perfectoid field, we use almost mathematics in the sense of [23]. We will denote morphism and equalities that may only exist after passing to the almost category by $\xrightarrow{a}$ and $\overset{a}{=}$.

If ϖ is any fixed pseudo-uniformiser of $\mathcal{O}_K$, we write $\log |K|$ for the set of ϵ such that $|\varpi|^\epsilon \in |K|$. For any such ϵ, when we write ϖ^ϵ and there is no obvious way to evaluate this otherwise, then we simply mean by this any choice of element in K such that $|\varpi^\epsilon| = |\varpi|^\epsilon$, and we implictly claim that our constructions are independent of this choice.

If R is any topological algebra, then by a weight κ over R we shall always mean a continuous group homomorphism $\kappa : \mathbb{Z}_p^\times \to R^\times$. Recall that the topological group $\mathbb{Z}_p^\times$ splits canonically into $\mathbb{F}_p^\times$ and a topologically free part. We let q be a fixed topological generator of the free part: Two natural choices would be $q = \exp(p)$ or $q = 1 + p$ for $p > 2$, and $q = \exp(4)$ or $q = 5$ for $p = 2$. Our constructions will be independent of this choice up to canonical isomorphism. For any weight κ over R, we define $T_\kappa := \kappa(q) - 1$. This determines κ up to a finite character, which will often be fixed. If R is a Banach algebra, we let $|T_\kappa| = |\kappa(q) - 1| = \max_{x \in \mathbb{Z}_p^\times} |\kappa(x) - 1|$, this is independent of our choice of q.

For any finite flat group scheme G over a base scheme S with identity section $e : S \to G$, we let $\omega_G = e^*\Omega^1_{G|S}$ be the conormal sheaf. We denote by $\mathrm{HT} : G^\vee \to \omega_G$ the Hodge–Tate morphism of sheaves $G^\vee \to \mathrm{Hom}(G, \mu_{p^\infty}) \to \omega_G$, where the last map sends $\varphi \mapsto \varphi^* \frac{d}{dt}$.

When discussing sheaves, we sometimes suppress pushforwards from notation when the base space and structure maps are clear from the context. For instance, when $f : X \to Y$ is a morphism of ringed spaces and G is a group that acts on X leaving f invariant, we sometimes abbreviate $(\mathcal{O}_X)^G := (f_*\mathcal{O}_X)^G$.

2 p-adic and perfectoid modular forms

Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $N \in \mathbb{N}$ be ≥ 3 coprime to p.

2.1 p-adic modular curves

Like in the introduction, let X be the compactified modular curve of tame level $\Gamma(N)$ over K. We denote by $X_{\mathbb{Z}_p}$ its canonical model over $\mathbb{Z}_p$. For any $\mathbb{Z}_p$-algebra R we denote by X_R the base change. We note that even though in the following we will frequently refer to [50], we deviate from Scholze's notation since we denote by X the compactified modular curve, while in Scholze's notation, X is the open modular curve and X^* is its compactification. We hope this does not cause any confusion – to our defence, this saves us roughly 1000 stars.

Let $\mathfrak{X}$ be the p-adic completion of $X_{\mathcal{O}_K}$ as a p-adic formal scheme. Let $\mathcal{X}$ be the adic analytification of X over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of [30], Proposition 3.8. Since X is proper, $\mathcal{X}$ can also be described as the adic generic fibre of $\mathfrak{X}$. Again, this deviates from the notation in [50] where $\mathcal{X}$ referred to open locus of good reduction, which can also be described as the adic generic fibre of the completion of the uncompactified modular curve.

Over X we have a universal semi-abelian scheme that we shall denote by $\pi : E \to X$. Let e be the identity section and let $\omega := e^*\Omega^1_{E|X}$ be the conormal sheaf on X. Let $\mathcal{E} \to \mathcal{X}$ be the analytification of E. We denote the analytification of ω on $\mathcal{X}$ by the same letter ω.

Definition 2.1.1. For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we consider the following subgroups of $\mathrm{GL}_2(\mathbb{Z}_p)$:

$$\begin{aligned}\Gamma_0(p^n) :=& \{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \bmod p^n\}\\ \Gamma_1(p^n) :=& \{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0, d \equiv 1 \bmod p^n\}.\end{aligned}$$

For $n < \infty$, we further denote by $\Gamma_1(p^n, \mathbb{Z}/p^n\mathbb{Z})$ the image of $\Gamma_1(p^n)$ in $\mathrm{GL}_2(\mathbb{Z}_p/p^n\mathbb{Z}_p)$.

For any $n \in \mathbb{N}$, we have over X modular curves $X_{\Gamma(p^n)} \to X_{\Gamma_1(p^n)} \to X_{\Gamma_0(p^n)} \to X$, with analytifications $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{\Gamma_1(p^n)} \to \mathcal{X}_{\Gamma_0(p^n)} \to \mathcal{X}$. Let us recall the moduli interpretation of these adic spaces and their transition maps: Away from the divisor $\partial \subseteq \mathcal{X}$ of cusps,

- the space $\mathcal{X} \backslash \partial$ represents the functor sending an adic space $S \to \mathrm{Spa}(K, \mathcal{O}_K)$ to the set of isomorphism classes of elliptic curves E over the ring $\mathcal{O}(S)$ together with a $\Gamma(N)$-structure, i.e. an isomorphism of group schemes $\underline{(\mathbb{Z}/N\mathbb{Z})}^2 \to E[N]$ ([27] Lemma 2.1).
- the space $\mathcal{X}_{\Gamma_0(p^n)} \backslash \partial \to \mathcal{X} \backslash \partial$ relatively represents the data of a cyclic subgroup scheme $D_n \subseteq E[p^n]$ of rank p^n defined over $\mathcal{O}(S)$.
- the space $\mathcal{X}_{\Gamma_1(p^n)} \backslash \partial \to \mathcal{X}_{\Gamma_0(p^n)} \backslash \partial$ relatively represents the data of an isomorphism $\alpha_1 : \underline{\mathbb{Z}/p^n\mathbb{Z}} \xrightarrow{\sim} D_n$ of $\mathcal{O}(S)$-group schemes.
- the space $\mathcal{X}_{\Gamma(p^n)} \backslash \partial \to \mathcal{X}_{\Gamma_1(p^n)} \backslash \partial$ relatively represents the choice of an isomorphism $\alpha : \underline{(\mathbb{Z}/p^n\mathbb{Z})}^2 \xrightarrow{\sim} E[p^n]$ such that the restriction $\alpha(-, 0)$ to the first factor equals α_1.

The group $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ has a natural left action on $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}$ which away from the cusps can be described in terms of moduli by letting $\gamma \in \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ act by sending $\alpha \mapsto \alpha \circ \gamma^\vee$ where $\gamma^\vee := (\det \gamma) \cdot \gamma^{-1}$. Here the dual is necessary to obtain a left-action. This group action restricts to natural group actions of $\Gamma_1(p^n, \mathbb{Z}/p^n\mathbb{Z})$ and $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{\Gamma_1(p^n)}$ and $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{\Gamma_0(p^n)}$, respectively, making these morphisms into finite étale torsors.

Let now $\frac{1}{2} > \epsilon \geq 0$ with $\epsilon \in \log |K|$. As explained in [50], §III.2, one may choose local lifts Ha of the Hasse invariant on local trivialisations of ω to make sense of the open subspace

$$\mathcal{X}(\epsilon) := \mathcal{X}(|\mathrm{Ha}| \geq |p|^\epsilon) \subseteq \mathcal{X}.$$

This analytic adic space has a canonical formal model $\mathfrak{X}(\epsilon) \to \mathfrak{X}$ that can be defined like in [50], Lemma III.2.13: Locally over any affine open $U = \mathrm{Spf}(R) = \mathfrak{X}$ where ω is trivial, choose a trivialisation and a lift Ha of the Hasse invariant. Then we set

$$\mathfrak{X}(\epsilon)|_U = \mathrm{Spf}(R\langle X\rangle/(X\mathrm{Ha} - p^\epsilon)).$$

This glues to give a flat formal scheme over $\mathcal{O}_K$ of topologically finite presentation.

2.1.1 The anticanonical locus

At level $\Gamma_0(p)$, the theory of the canonical subgroup [34] leads to a decomposition

$$\mathcal{X}_{\Gamma_0(p)}(\epsilon) = \mathcal{X}_{\Gamma_0(p)}(\epsilon)_c \sqcup \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$$

into two disjoint open components, namely the canonical one, denoted by $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_c$, for which the projection to $\mathcal{X}(\epsilon)$ is an isomorphism, and the "anticanonical" one, for which the projection has degree p. In the following, we will always work with the anticanonical locus. In doing so, we are following [50] and [3], but this is the point where our discussion deviates from the construction of p-adic modular forms via perfectoid modular curves in [17].

We denote by $\mathcal{X}_{\Gamma_1(p^n)}(\epsilon)_a$ the pullback of $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \subseteq \mathcal{X}_{\Gamma_0(p)}$ to $\mathcal{X}_{\Gamma_1(p^n)}$, and similarly for $\mathcal{X}_{\Gamma(p^n)}$. We then have a tower

$$\mathcal{X}_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_1(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \tag{2}$$

where the first morphism is finite étale Galois with group $\Gamma_1(p^n, \mathbb{Z}/p^n\mathbb{Z})$ and the second morphism is finite étale Galois for the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. The third morphism, however, is finite flat but *not* étale, due to ramification over the cusps.

2.1.2 The Atkin–Lehner isomorphism

The Atkin–Lehner isomorphism is the map which can on the level of moduli be described as

$$\psi_n : \mathcal{X}(p^{-n}\epsilon) \xrightarrow{\sim} \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a, \quad E \mapsto (E/H_n(E), E[p^n]/H_n(E))$$

where $H_n(E)$ denotes the canonical subgroup. Its inverse is given by sending $(E', D_n) \mapsto E'/D_n$. Via ψ_n, we can in particular think of the formal model $\mathfrak{X}(p^{-n}\epsilon)$ of $\mathcal{X}(p^{-n}\epsilon)$ as being a formal model $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a := \mathfrak{X}(p^{-n}\epsilon)$ of $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$. Via ψ_n and ψ_{n-1}, we can moreover identify the forgetful morphism $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^{n-1})}(\epsilon)$ with a morphism

$$\phi : \mathcal{X}(p^{-n}\epsilon) \to \mathcal{X}(p^{n-1}\epsilon).$$

The following result is proved as part of Theorem III.2.15 in [50] in the case of $K = \mathbb{Q}_p^{\mathrm{cyc}}$. We shall deduce the following version from a more general statement in section §6.

Lemma 2.1.2. *The morphism $\phi : \mathcal{X}(p^{-n}\epsilon) \to \mathcal{X}(p^{-(n-1)}\epsilon)$ has a canonical finite flat formal model $\phi : \mathfrak{X}(p^{-n}\epsilon) \to \mathfrak{X}(p^{-(n-1)}\epsilon)$. It reduces to the relative Frobenius map mod $p^{1-\epsilon}$.*

We may interpret ϕ as a formal model $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathfrak{X}_{\Gamma_0(p^{n-1})}(\epsilon)_a$ of the forgetful map.

2.1.3 Infinite level modular curves

By [50], Corollary III.2.19, there is an affinoid perfectoid space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ such that

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a.$$

This is constructed as the generic fibre of the formal scheme limit

$$\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a := \varprojlim \mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a = \varprojlim_{\phi} \mathfrak{X}(p^{-n}\epsilon),$$

and the idea is that $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ is perfectoid because ϕ reduces to the relative Frobenius.

For varying $n \in \mathbb{N}$, the towers from (2) fit into an inverse system for the respective forgetful maps. Away from the cusps, this defines an element in the pro-étale site of $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$ in the sense of [49]. Over the boundary, this is still an object in the quasi-pro-étale site [51].

In particular, away from the divisor ∂ of cusps, the tower of spaces $(\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a\backslash\partial)_{n\in\mathbb{N}}$ is a perfectoid cover of $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a\backslash\partial$ in the pro-étale site $(\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a\backslash\partial)_{\text{proét}}$. Similarly for $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ and $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$. In the limit $n \to \infty$, we thus obtain a tower of torsors of perfectoid spaces over $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$ with Galois groups summarised in the following diagram:

$$\begin{tikzcd}
\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r] \arrow[r, bend right, dash, "\Gamma_1(p^\infty)"'] \arrow[rr, bend right=50, dash, "\Gamma_0(p^\infty)"'] \arrow[rrr, bend right=70, dashed, dash, "\Gamma_0(p)"'] & \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \arrow[r] \arrow[r, bend right, dash, "\mathbb{Z}_p^\times"'] & \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \arrow[r] & \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \arrow[r] & \mathcal{X}(\epsilon)
\end{tikzcd}$$

Here the first two morphisms are pro-étale, but the third morphism is only quasi-proétale, and becomes pro-étale away from the boundary. The composition $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}(\epsilon)$, on the other hand, is not Galois, even away from the boundary. In order to obtain a Galois torsor, we therefore will have to replace $\mathcal{X}$ in the following by the space $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$, for which $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$ is a pro-étale $\Gamma_0(p)$-torsor away from the cusps. This is the cover we are going to use for the definition of modular forms. In particular, in our setting, the automorphic bundles will live over $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$, although we can of course always identify this with $\mathcal{X}(p^{-1}\epsilon)$ by way of the Atkin–Lehner isomorphism.

Finally, we recall from [50], §III.3 the $\Gamma_0(p)$-equivariant Hodge–Tate period morphism

$$\pi_{\text{HT}} : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathbb{P}^{1,\text{an}}_K$$

whose moduli interpretation in terms of the Hodge–Tate map gives rise to a canonical isomorphism $q^*\omega = \pi^*_{\text{HT}}\mathcal{O}(1)$ where q is the projection $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}$.

The following Lemma will be useful for the construction of modular forms:

Lemma 2.1.3 ([10], Proposition 3.7)**.** *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.*

1. $(\mathcal{O}^+_{\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a})^{\Gamma_0(p^n)} = \mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a}$
2. $(\mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a})^{\mathbb{Z}_p^\times} = \mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}$.
3. $(\mathcal{O}^+_{\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a})^{\Gamma_1(p^\infty)} = \mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}$

Proof. For the case of $n < \infty$, we first remove the cusps and consider the smooth rigid space $Y := \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a\backslash\partial$. For this one can essentially argue like in [17], Lemma 2.26, except for a difference in base field: We work in the pro-étale site $Y_{\text{proét}}$ of Y in the sense of [49] and use the completed structure sheaf $\widehat{\mathcal{O}}^+_Y$. The $\Gamma_0(p^n)$-torsor property of $Y_\infty := \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a\backslash\partial \to Y$ then shows that $(\widehat{\mathcal{O}}^+_{Y_\infty})^{\Gamma_0(p^n)} = \widehat{\mathcal{O}}^+_Y$. To deduce part 1 away from the cusps, we use that the morphism of sites $v : Y_{\text{proét}} \to Y_{\text{an}}$ induces an isomorphism $v_*\widehat{\mathcal{O}}^+_{Y_{\text{proét}}} = \mathcal{O}^+_Y$ by [38], Theorem 8.2.3. We refer to [10], Proposition 3.7 for a proof that one can extend to the cusps.

For part 2, we use that $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ is a pro-étale $\mathbb{Z}_p^\times$-torsor of perfectoid spaces: The torsor property in the pro-étale site of $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ in the sense of [51] gives the desired result since the integral pro-étale structure presheaf $\mathcal{O}^+$ on $(\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a)_{\text{proét}}$ is a sheaf by Proposition 8.5.(iii), [51]. The same argument proves parts 3 and 1 for $n = \infty$. $\square$

2.2 p-adic modular forms via the anticanonical locus

In this section, we give the construction of overconvergent p-adic modular forms we work with. The definition we work with is the one from [10], which is essentially a hybrid between the definitions of [17] and [3]: We use the perfectoid modular curve at infinite level and the Hodge–Tate period map like the former, but instead of the canonical locus, we use the anticanonical locus and the anticanonical tower, essentially like the latter, in the guise of an Igusa tower. The reason we do this is that for the tilting equivalence we need to work with the anticanonical locus, where we have a canonical tilting isomorphism of modular curves.

2.2.1 Analytic continuations of weights and the Hodge-Tate period map

Definition 2.2.1.
1. Let $\mathbb{G}_a^{\mathrm{an}}$ be the adic analytification of the scheme $\mathbb{G}_{a,K} = \mathbb{A}^1_K$. This is the adic group representing the functor that sends an adic space Z over K to $\mathcal{O}(Z)$.
2. Let $\hat{\mathbb{G}}_a$ be the adic generic fibre of the p-adic completion of the scheme $\mathbb{G}_{a,\mathcal{O}_K} = \mathbb{A}^1_{\mathcal{O}_K}$. This is the closed unit disc around 0 in $\mathbb{G}_a^{\mathrm{an}}$. It represents the functor that sends an adic space Z over K to $\mathcal{O}^+(Z)$.
3. Let $\mathbb{G}_m^{\mathrm{an}}$ be the adic analytification of the scheme $\mathbb{G}_{m,K}$. It represents the functor that sends an adic space Z over K to $\mathcal{O}(Z)^\times$.
4. Let $\hat{\mathbb{G}}_m$ be the adic generic fibre of the p-adic completion of $\mathbb{G}_{m,\mathcal{O}_K}$. This is the closed unit disc around 1 in $\mathbb{G}_m^{\mathrm{an}}$. It represents the functor that sends an adic space Z over K to $\mathcal{O}^+(Z)^\times$.

Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a p-adic weight, i.e. a continuous group homomorphism. One can always find an analytic continuation of κ, by which we mean the following: Let $\underline{\mathbb{Z}_p^\times}$ be the profinite perfectoid group over K associated to $\mathbb{Z}_p^\times$. Explicitly,

$$\underline{\mathbb{Z}_p^\times} = \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K), \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K)).$$

Then we can interpret κ geometrically as a morphism of adic groups $\kappa : \underline{\mathbb{Z}_p^\times} \to \hat{\mathbb{G}}_m$ by the universal property of $\hat{\mathbb{G}}_m$. We recover the group homomorphism by taking K-points.

For any $\delta > 0$, denote by $\underline{\mathbb{Z}_p^\times}(\delta)$ the thickening of $\underline{\mathbb{Z}_p^\times}$ inside $\hat{\mathbb{G}}_m$ given by the union of balls $\{x \in \hat{\mathbb{G}}_m \mid |x - a| \leq \delta\}$ for all $a \in \hat{\mathbb{G}}_m(\mathbb{Z}_p) \subseteq \mathbb{G}_m^{\mathrm{an}}(\mathbb{Z}_p)$. We similarly define $\underline{\mathbb{Z}_p}(\delta)$ to be the thickening of $\hat{\mathbb{G}}_a(\mathbb{Z}_p) \subseteq \mathbb{A}^{1,\mathrm{an}}$ of radius δ. Our δ plays the role played by "w" in [17], and the precise relation is $\delta = |p|^w$. The reason we choose to work with δ is that this allows us to use notation in analogy with the parameter ϵ we use throughout for overconvergent neighbourhoods of $\mathcal{X}$, whereas [17] work with a different system of neighbourhoods.

For $\delta > 0$ small enough, there always exists a unique extension of κ to a morphis

$$\kappa^{\mathrm{an}} : \underline{\mathbb{Z}_p^\times}(\delta) \to \hat{\mathbb{G}}_m$$

of adic spaces that extends the morphism $\kappa^{\mathrm{an}} : \underline{\mathbb{Z}_p^\times} \to \hat{\mathbb{G}}_m$, see [13] Proposition 8.3.

For any such $\delta > 0$ assume for a moment that we have $\epsilon > 0$ such that

$$\pi_{\mathrm{HT}}(\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a) \subseteq \underline{\mathbb{Z}_p}(\delta) \subseteq \hat{\mathbb{G}}_a \subseteq \mathbb{A}^{1,\mathrm{an}} \subseteq \mathbb{P}^{1,\mathrm{an}}.$$

Let $\mathfrak{z}$ be the function on $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ defined by pullback along π_{HT} of the canonical coordinate z on the chart $\mathbb{A}^1 \subseteq \mathbb{P}^1$ around $(0:1)$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ we can define an invertible function on $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ by

$$\kappa(c\mathfrak{z} + d) : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \xrightarrow{\pi_{\mathrm{HT}}} \underline{\mathbb{Z}_p}(w) \xrightarrow{z \mapsto cz+d} \underline{\mathbb{Z}_p^\times}(p^{-1}w) \xrightarrow{\kappa^{\mathrm{an}}} \hat{\mathbb{G}}_m.$$

This is our analogue of the factor of automorphy $(cz + d)^k$ for complex modular forms.

We would like a precise condition on when our assumption on ϵ is satisfied:

Lemma 2.2.2. *1. Let $s \in \mathbb{N}$ and $\epsilon \leq \frac{1}{3p^s(p-1)}$. Then $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) \subseteq \mathbb{Z}_p(p^{-s})$.*

2. Assume moreover that $|T_\kappa| \leq |p|^{1/p^s(p-1)}$. Then for any $c \in p\mathbb{Z}_p$, $d \in \mathbb{Z}_p^\times$, the function $\kappa(c\mathfrak{z}+d)$ on $\mathcal{X}_{\Gamma(p^\infty)}(0)_a$ extends uniquely to a function on $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$.

Remark 2.2.3. Conversely, it is also true that for any $\frac{1}{2} > \epsilon > 0$, there is $s \in \mathbb{N}$ such that $\mathbb{Z}_p(p^{-s}) \subseteq \pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a)$. This follows from [50], Lemma III.3.8 since $\mathbb{Z}_p(p^{-s})$ is quasi-compact (see also [17], Theorem 2.17.(3)). In particular, the open subspaces $\pi^{-1}_{\mathrm{HT}}(\mathbb{Z}_p(p^{-w}))$, which are the analogues of the neighbourhoods "$\mathcal{X}_{\infty,w}$" used by Chojecki–Hansen–Johansson to define modular forms, are cofinal in the family of neighbourhoods $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ that we use.

Proof. We first explain how to deduce 2 from 1: By [17], Proposition 2.6, the assumptions on s ensure that κ extends to $\kappa^{\mathrm{an}} : \mathbb{Z}_p^\times(p^{-(s+1)}) \to \hat{\mathbb{G}}_m$. We note that the definition of "weight" in [17] technically does not allow weights valued in K, but the proof of the Proposition goes through verbatim since $\mathcal{O}(\mathbb{Z}_p^\times(p^{-(s+1)}))$ is a uniform $\mathbb{Q}_p$-Banach algebra. If we can prove part 1, then $\mathfrak{z}$ sends $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ into $\mathbb{Z}_p(p^{-s})$, and thus $c\mathfrak{z}+d$ has image in $\mathbb{Z}_p^\times(p^{-(s+1)})$. The two functions then compose to give the desired invertible function $\kappa(c\mathfrak{z}+d)$.

We are left to prove the first part: It is asserted in [17], §2.7 that this follows from [3], Proposition 3.2.1. Let us elaborate on this, and in particular on how to deal with the case of $p = 2$ which is not covered by the cited Proposition.

It suffices to check this statement on (C, C^+)-points. Let $x \in \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a(C, C^+)$. The case of the boundary is clear, so we may assume that x corresponds to an elliptic curve $E|C$ with anticanonical trivialisation $\alpha : \mathbb{Z}_p^2 \to T_pE$ such that $|\mathrm{Ha}(E)| \geq |p|^\epsilon$. Let L be the kernel of the integral Hodge–Tate-map, i.e. there is a left exact sequence

$$0 \to L \to T_pE \otimes \mathcal{O}_C \xrightarrow{\mathrm{HT}} \omega_E.$$

Then by definition it is clear that $L \subseteq T_pE \otimes \mathcal{O}_C$ is saturated. With respect to α, we can therefore write L as a point $(a : b) \in \mathbb{P}^1(\mathcal{O}_C)$ with a, b coprime.

Let now $n = s+1$, then our conditions on ϵ ensure that there is a canonical subgroup $H_n \subseteq E[p^n]$ of rank p^n. Upon reduction mod p^n, we get an injection $L/p^n \to E[p^n] \otimes_{\mathbb{Z}} \mathcal{O}_C$ which fits into a not necessarily exact complex

$$0 \to L/p^n \to E[p^n] \otimes \mathcal{O}_C \to \omega_{E[p^n]}.$$

Let moreover $x = n - \frac{p^n}{p-1}\epsilon$ and $y = n - \frac{p^n-1}{p-1}\epsilon$. We claim that we then have $L/p^x = H_n \otimes \mathcal{O}_C/p^x$ inside $E[p^n] \otimes \mathcal{O}_C/p^x$. This would prove the Lemma, since $H_n \otimes_{\mathbb{Z}} \mathcal{O}_C/p^x \subseteq E[p^n] \otimes \mathcal{O}_C/p^x$ clearly has rational coordinates, and thus $(a : b)$ reduces to a point in the image of $\mathbb{P}^1(\mathbb{Z}_p)$ in $\mathbb{P}^1(\mathcal{O}_C/p^x)$, which proves $a/b \in \mathbb{Z}_p + p^x\mathcal{O}_C$. Since $x > n-1 = s$ by our assumptions on ϵ, this implies $\pi_{\mathrm{HT}}(x) \in \mathbb{Z}_p(p^{-s})$ as desired.

To see that $L/p^x = H_n \otimes \mathcal{O}_C/p^x$, we use [1], Proposition 3.2.2. The first part says that $\omega_{E[p^n]} \otimes \mathcal{O}_C/p^y = \omega_{H_n} \otimes \mathcal{O}_C/p^y$. We note that Proposition in [1] is only stated for $p \neq 2$, but this assumption is not essential: For the first part, we can in the proof replace [1], Theorem 3.1.1 by [3], Corollaire A.2.4. The Hodge-Tate map can therefore mod p^y be described as

$$\mathrm{HT}_y : E[p^n] \otimes \mathcal{O}_C/p^y \to \omega_{E[p^n]} \otimes \mathcal{O}_C/p^y = \omega_{H_n} \otimes \mathcal{O}_C/p^y.$$

By the second part of [1] Proposition 3.2.2 (also cf [46] Proposition 3.1) we have $\operatorname{im}\mathrm{HT}_y \cong \mathcal{O}_C/p^x$. This is proved by reducing to the case $n = 1$, where the statement follows by Oort–Tate theory, see [22], §6.5 Lemme 9. In particular, this also works in the case of $p = 2$.

Let now $L' := \ker \mathrm{HT}_y$, then by additivity of degrees in short exact sequences of $\mathcal{O}_C$-modules, we conclude that $\deg L' = 2y - x = y + \partial$ where $\partial = \epsilon/(p-1)$. Since $L/p^y \subseteq L'$, and L' is p^y-torsion, this shows that $L' \cong \mathcal{O}_C/p^y \oplus \mathcal{O}_C/p^\partial = \mathcal{O}_C/p^y \oplus p^x\mathcal{O}_C/p^y$.

Passing to the quotient $\mathrm{HT}_x : E[p^n] \otimes \mathcal{O}_C/p^x \to \omega_{H_n} \otimes \mathcal{O}_C/p^x$, we conclude that the image L'_x of L' in $E[p^n] \otimes \mathcal{O}_C/p^x$ is a subgroup of degree x. But $H_n \otimes \mathcal{O}_C/p^x$ is contained in L'_x and has this very degree. The same is true for L/p^x. Thus $L/p^x = L'_x = H_n \otimes \mathcal{O}_C/p^x$. $\square$

2.2.2 Definition of p-adic overconvergent modular forms

Definition 2.2.4.

1. For any weight $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$, we let $\epsilon_\kappa^{\mathrm{def}} := \frac{1}{3p^s(p-1)}$ where s is the smallest integer such that $|T_\kappa| \le |p|^{1/p^s(p-1)}$ (we recall $|T_\kappa| := \max_{x \in \mathbb{Z}_p^\times} |\kappa(x) - 1|$).

2. For any weight $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$, we let $\epsilon_\kappa > 0$ be defined implicitly by the equation $|\alpha|^{1/p^{r_0+1}} = |p|^{\epsilon_\kappa}$, where $r_0 := 5$ if $p = 2$ and $r_0 := 3$ if $p > 2$, and $|\alpha| := \max(|T_\kappa|, |p|)$.

These technical conditions serve the following purpose: If $\epsilon < \epsilon_\kappa^{\mathrm{def}}$, then Lemma 2.2.2 says that we can define the function $\kappa(c\mathfrak{z} + d)$, and consequently the sheaf of modular forms, as we shall do instantly. Since $3\epsilon_\kappa^{\mathrm{def}}/p^{r_0} > \epsilon_\kappa$, the condition $\epsilon \le \epsilon_\kappa$ is substantially stronger, and ensures this sheaf is invertible. We do not claim that ϵ_κ is maximal with this property, but it is as large as possible for our constructions in the later chapters to work out.

Definition 2.2.5. Let $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight. Let $0 \le \epsilon \le \epsilon_\kappa^{\mathrm{def}}$, or $\epsilon \le p^{-1}\epsilon_\kappa^{\mathrm{def}}$ if $n = 0$. For $n \ge 1$, let $q : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ be the forgetful map.

1. For $n \ge 1$, we define a sheaf $\omega^\kappa_{\Gamma_0(p^n)} \subseteq q_*\mathcal{O}_{\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a}$ on $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ by setting
$$\omega^\kappa_{\Gamma_0(p^n)}(U) = \Big\{ f \in q_*\mathcal{O}_{\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a}(U) \,\Big|\, \gamma^* f = \kappa^{-1}(c\mathfrak{z}+d) f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^n) \Big\}.$$
We sometimes also call this $\omega^\kappa_{\Gamma_0(p^n)}(\epsilon)$, but usually suppress ϵ from notation. For $n = 0$, we use the Atkin–Lehner isomorphism $\psi : \mathcal{X}(\epsilon) \xrightarrow{\sim} \mathcal{X}_{\Gamma_0(p)}(p\epsilon)_a$ of §2.1.2 to set
$$\omega^\kappa := \omega^\kappa_{\Gamma_0(1)} := \psi^* \omega^\kappa_{\Gamma_0(p)}(p\epsilon).$$

2. For any $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$, we let $M_{\kappa, \Gamma_0(p^n)}(\epsilon) := \Gamma(\mathcal{X}_{\Gamma_0(p^n)}(\epsilon), \omega^\kappa_{\Gamma_0(p^n)})$. As we shall discuss in §2.7-§2.9, this is the K-Banach space of p-adic modular forms of weight κ, tame level $\Gamma(N)$, wild level $\Gamma_0(p^n)$ and radius of overconvergence ϵ. We moreover let
$$M_\kappa(\epsilon) = \Gamma(\mathcal{X}(\epsilon), \omega^\kappa).$$

3. By replacing $\mathcal{O}$ with $\mathcal{O}^+$ in the above definition, we obtain the integral subsheaves $\omega^{\kappa,+}_{\Gamma_0(p^n)} \subseteq \omega^\kappa_{\Gamma_0(p^n)}$ and $\omega^{\kappa,+} := \omega^{\kappa,+}_{\Gamma_0(1)} := \psi^* \omega^{\kappa,+}_{\Gamma_0(p)}(p\epsilon) \subseteq \omega^\kappa$. They define the integral subspaces $M^+_{\kappa,\Gamma_0(p^n)}(\epsilon) := \Gamma(\mathcal{X}_{\Gamma_0(p^n)}(\epsilon), \omega^{\kappa,+}_{\Gamma_0(p^n)}) \subseteq M_{\kappa,\Gamma_0(p^n)}(\epsilon)$ and $M^+_\kappa(\epsilon) \subseteq M_\kappa(\epsilon)$.

4. We set $\omega^{\kappa,\mathrm{perf}} := \omega^\kappa_{\Gamma_0(p^\infty)}$ and $M^{\mathrm{perf}}_\kappa(\epsilon) := M_{\kappa,\Gamma_0(p^\infty)}(\epsilon)$. These are the sheaf and K-vector space of perfectoid p-adic overconvergent modular forms, respectively. We define the integral perfectoid modular forms $M^{+,\mathrm{perf}}_\kappa(\epsilon)$ using $\omega^{\kappa,+,\mathrm{perf}} := \omega^{\kappa,+}_{\Gamma_0(p^\infty)}$.

5. It is clear that $M_{\kappa,\Gamma_0(p^n)}(\epsilon) \subseteq M_{\kappa,\Gamma_0(p^m)}(\epsilon)$ for any $n \le m \in \mathbb{Z}_{\ge 1} \cup \{\infty\}$. This extends to $n = 0$, since we will see in Proposition 2.4.1.3 below that there is also a natural adjunction morphism $M_\kappa(\epsilon) \hookrightarrow M_{\kappa,\Gamma_0(p^m)}(\epsilon)$ compatible with the above inclusions. In particular, every modular form in $M_{\kappa,\Gamma_0(p^n)}(\epsilon)$ can be regarded as a perfectoid modular form. If we wish to emphasize that a perfectoid modular form is already contained in $M_{\kappa,\Gamma_0(p^n)}(\epsilon)$ for some $n < \infty$, we shall call it a "true p-adic modular form".

Conversely, we will prove in Corollary 4.2.5 below that there is a way to go back from perfectoid modular forms to true p-adic modular forms for $\Gamma_0(p^n)$ by way of trace maps:

Proposition 2.2.6. *Let $\epsilon \leq \epsilon_\kappa$. Then $M^+_{\kappa,\Gamma_0(p^n)}(\epsilon) \hookrightarrow M^{+,\mathrm{perf}}_{\kappa}(\epsilon)$ has an $\mathcal{O}_K$-linear section*

$$\mathrm{tr} : M^{+,\mathrm{perf}}_{\kappa}(\epsilon) \to p^{-3\epsilon/p^{n-3}(p-1)} M^+_{\kappa,\Gamma_0(p^n)}(\epsilon).$$

We note that we understand the field of definition we work over to be implicit in the variable κ. In particular, we still use this notation when we later work in characteristic p.

We will prove in Proposition 2.7.6 below:

Proposition 2.2.7. *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\epsilon \leq \epsilon_\kappa$. Then $\omega^{\kappa,+}_{\Gamma_0(p^n)}$ is an invertible $\mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a}$-module. In particular, $\omega^{\kappa}_{\Gamma_0(p^n)}$ is an invertible $\mathcal{O}_{\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a}$-module.*

2.3 q-expansions

There is a good theory of q-expansions of modular forms in this setting. While we will not need it for the proofs of any of our main results, we shall include a brief summary since we hope it helps to illustrate the relation between p-adic and perfectoid modular forms.

In parallel to the classical theory of q-expansions, where q-expansions arise from cusps of the compactification of the upper half plane or the compactified modular curve $\mathcal{X}$, the q-expansions in our setting arise from the cusps of the perfectoid space $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$. We briefly outline some facts about the cusps of this space, and refer to [27] for details.

For simplicity, let us in this section assume that K contains a primitive N-th root of unity. This condition is not in any way essential, but simplifies the exposition. In particular, whenever we mention q-expansions below for expository purposes, we will tacitly assume that K contains a primitive N-th root of unity.

Around any cusp c of the finite level modular curve $\mathcal{X}$, there is then an open immersion $D \hookrightarrow \mathcal{X}$ of the open unit disc $D = \mathrm{Spf}(\mathcal{O}_K[[q]])^{\mathrm{ad}}_\eta$ to an open neighbourhood of the cusp which sends the origin of D to the cusp. We can see D as a parameter space for Tate curves $\mathrm{T}(q)$ equipped with tame level structure corresponding to c.

The same discussion applies to the modular curves $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ for any n, whose cusps can be identified with those of $\mathcal{X}$ via the forgetful morphism. We choose to make the following normalisation convention, which described the transition maps at the cusps in the anticanonical locus: For $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$, we renormalise the discs to be given by the disc $D_n = \mathrm{Spf}(\mathcal{O}_K[[q^{1/p^n}]])^{\mathrm{ad}}_\eta$ in the parameter q^{1/p^n} rather than q (cf [27], Proposition 2.10). This is chosen so that the base space $\mathcal{X}(\epsilon)$ has q-expansions in $\mathcal{O}_K[[q]]$.

When we now consider the tower of modular curves with infinite level structures, we obtain a tower of Cartesian squares of vertical open immersions (Theorem 3.18, [27]):

$$\begin{tikzcd}
\underline{\Gamma_0(p^\infty)} \times D_\infty \arrow[r] \arrow[d, hook] & \underline{\mathbb{Z}_p^\times} \times D_\infty \arrow[r] \arrow[d, hook] & D_\infty \arrow[r] \arrow[d, hook] & D_n \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\
\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r] & \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \arrow[r] & \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \arrow[r] & \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \arrow[r] & \mathcal{X}(\epsilon).
\end{tikzcd} \tag{3}$$

where the morphism on the top left is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$. Here the space D_∞ is the perfectoid open unit disc and arises as the tilde-limit $D_\infty \sim \varprojlim D_n$. More explicitly, it is the open subspace of the closed perfectoid unit disc $\mathrm{Spa}(K\langle q^{1/p^\infty}\rangle, \mathcal{O}_K\langle q^{1/p^\infty}\rangle)$ where $|q| < 1$. By specialisation at $a \in \mathbb{Z}_p^\times$, we get for any cusp c of $\mathcal{X}$ a locally closed immersion

$$\gamma_c : D_\infty \hookrightarrow \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$$

that sends the origin to c. We note that at level $\Gamma_1(p^\infty)$ this is not an open immersion anymore, essentially because $\underline{\mathbb{Z}_p^\times}$ carries the profinite topology rather than the discrete one.

While the functions $\mathcal{O}(D)$ and $\mathcal{O}(D_\infty)$ are subrings of formal power series of $K[[q]]$ and $K[[q^{1/p^\infty}]]$ with convergence conditions, the rings of bounded functions are simply

$$\mathcal{O}^+(D) = \mathcal{O}_K[[q]] \text{ and } \mathcal{O}^+(D_\infty) = \mathcal{O}_K[[q^{1/p^\infty}]].$$

We thus obtain q-expansions of modular forms as usual, just that in general these are now power series in q^{1/p^∞} rather than just power series in q. Alternatively, we can consider profinite families of q-expansions defined by functions on the space $\mathbb{Z}_p^\times \times D_\infty$ where we have

$$\mathcal{O}^+(\mathbb{Z}_p^\times \times D_\infty) = \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]]).$$

Definition 2.3.1. Let f be a function on $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$. Then for any cusp $c \in \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$, we define the q-expansion of f at c to be the image of f under the morphism

$$\gamma_c : \mathcal{O}(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a) \to \mathcal{O}_K[[q^{1/p^\infty}]][1/p]$$

corresponding to the map $\gamma_c : D_\infty \hookrightarrow \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$. We shall denote this image by f_c.

Given a cusp $c_0 \in \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$, we define the q-expansion f_{c_0} of f at c_0 to be the image of

$$\gamma_{c_0} : \mathcal{O}(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a) \to \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]])[1/p].$$

In diagram (3), the morphism $\mathbb{Z}_p^\times \times D_\infty \hookrightarrow \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ is in fact canonical after a choice of a compatible system of primitive unit roots ζ_{p^n} in $\mathbb{Q}_p^{\mathrm{cyc}}$. In particular, for any cusp $c_0 \in \mathcal{X}$, we can speak of a "canonical" cusp c over c_0 defined by the image of $0 \in \mathbb{Z}_p^\times$. This together with the following Lemma often reduces us to considering cusps of $\mathcal{X}$.

Lemma 2.3.2. *Let c_0 be a cusp of $\mathcal{X}$. Let c be a cusp of $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ over c_0. Let $\gamma \in \mathbb{Z}_p^\times$, giving another cusp γc over c_0. Then for any weight $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ and any $f \in M_\kappa^{+,\mathrm{perf}}(\epsilon)$,*

$$f_{\gamma c} = \kappa^{-1}(\gamma) f_c.$$

In particular, up to a scalar, the q-expansion of f at c only depends on c_0.

Proof. This follows from the $\mathbb{Z}_p^\times$-equivariance in [27], Theorem 3.1: The q-expansion of $\gamma^* f$ at c equals the q-expansion of f at γc. Thus $\kappa^{-1}(\gamma) f_c = (\kappa^{-1}(\gamma) f)_c = (\gamma^* f)_c = f_{\gamma c}$. □

It therefore often makes sense to talk about "the" q-expansion of f at $c_0 \in \mathcal{X}$, when the statement does not depend on a scaling of f by a unit in $\mathcal{O}_K^\times$.

The following is a perfectoid version of Katz' q-expansion principles, Theorem 1.3.1 [34]:

Proposition 2.3.3 (q-expansion principle I, [27], Proposition 6.1)**.** *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight and let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then for $f \in M_{\kappa,\Gamma_0(p^n)}(\epsilon)$, the following are equivalent:*

1. *$f = 0$.*
2. *The q-expansion $f_c \in \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]])[1/p]$ vanishes at all cusps c.*

Proposition 2.3.4 (q-expansion principle II, [27], Corollary 6.10.)**.** *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight and let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then for $f \in M_\kappa^{\mathrm{perf}}(\epsilon)$, the following are equivalent:*

1. *f is already contained in $M_{\kappa,\Gamma_0(p^n)}(\epsilon)$.*
2. *The q-expansion f_c is already in $\mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^n}]])[1/p]$ at all cusps c.*

2.4 The Atkin–Lehner isomorphism

2.4.1 Frobenius lifts

In the classical theory of p-adic modular forms, one usually does not define p-adic modular forms of $\Gamma_0(p^n)$-level, since these can be canonically interpreted as modular forms of tame level by an Atkin–Lehner isomorphism. For our considerations, however, it will be important to actually make a difference between the two. This can be understood in terms of a normalisation of q-expansions, as we shall now explain.

2.4.2 The Atkin–Lehner isomorphism of modular forms

We have already introduced in §2.1.2 the Atkin–Lehner isomorphism

$$\psi^n : \mathcal{X}(p^{-n}\epsilon) \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a.$$

Since our base space is $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$, not $\mathcal{X}(\epsilon)$, we will also need the version for $\Gamma_0(p)$, which we shall also denote by ψ and which we may define as the composition

$$\psi^n : \mathcal{X}_{\Gamma_0(p)}(p^{-n}\epsilon)_a \xrightarrow{\psi^{-1}} \mathcal{X}(p^{-(n+1)}\epsilon)_a \xrightarrow{\psi^{n+1}} \mathcal{X}_{\Gamma_0(p^{n+1})}(\epsilon)_a \tag{4}$$

Composing with the forgetful map λ^n, we obtain the "Frobenius lift"

$$\phi^n : \mathcal{X}_{\Gamma_0(p)}(p^{-n}\epsilon)_a \overset{\psi^n}{\underset{\sim}{\longrightarrow}} \mathcal{X}_{\Gamma_0(p^{n+1})}(\epsilon)_a \xrightarrow{\lambda^n} \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \tag{5}$$

We will need this later to define the U_p-operator.

Proposition 2.4.1. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K$ be any weight. Let $\epsilon < p^{-1}\epsilon_\kappa$.*

1. *The action of $\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{X}_{\Gamma(p^\infty)}$ restricts to a map $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma(p^\infty)}(p^{-n}\epsilon)_a$ which defines an isomorphism of sheaves on $\mathcal{X}(p^{-n}\epsilon)$*

$$\phi^{n*}\omega^\kappa = \psi^{n*}\omega^\kappa_{\Gamma_0(p^n)} = \iota^{n*}\omega^\kappa$$

   where $\iota^n : \mathcal{X}_{\Gamma_0(p)}(p^{-n}\epsilon)_a \hookrightarrow \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$ is the inclusion map.

2. *The adjoint map $\omega^\kappa_{\Gamma_0(p^n)} \to \psi^n_*\iota^{n*}\omega^\kappa$ defines on global sections an isomorphism*

$$\mathrm{AL}^n : M_{\kappa,\Gamma_0(p^n)}(\epsilon) \xrightarrow{\sim} M_\kappa(p^{-n}\epsilon),$$

   which on q-expansions is given by sending $\sum_{m=0}^\infty a_m q^{m/p^n} \mapsto \sum_{m=0}^\infty a_m q^m \in K[[q]]$.

3. *There is a natural isomorphism $\lambda^{n*}\omega^\kappa = \omega^\kappa_{\Gamma_0(p^n)}$ whose adjoint gives an inclusion map*

$$M_\kappa(\epsilon) \hookrightarrow M_\kappa(p^{-n}\epsilon),$$

   which on q-expansions is the inclusion $K[[q]] \hookrightarrow K[[q^{1/p^n}]]$.

Proof. We postpone the proof to . □

Remark 2.4.2. Proposition 2.4.1.2 says that $M_{\kappa,\Gamma_0(p^n)}(\epsilon) \cong M_\kappa(p^{-n}\epsilon)$ can be canonically identified, and thus in this sense modular forms of level $\Gamma_0(p^n)$ give "nothing new". However, the statements about q-expansions make it clear that the canonical isomorphism does *not* commute with the inclusions $M_\kappa(\epsilon) \subseteq M_{\kappa,\Gamma_0(p^n)}(\epsilon)$ and the restriction $M_\kappa(\epsilon) \subseteq M_\kappa(p^{-n}\epsilon)$, since the former is given on q-expansions by $\sum a_n q^n \mapsto \sum a_n q^n$. We will therefore throughout distinguish $\omega^\kappa_{\Gamma_0(p^n)}$ and $\iota^{n*}\omega^\kappa$ by their canonical adjunction maps from ω^κ.

This is in line with our normalisation in diagram (3) saying that functions over $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ have q-expansions in $K[[q^{1/p^n}]]$, as do modular forms in $M_{\kappa,\Gamma_0(p^n)}(\epsilon)$.

2.5 Hecke operators

2.5.1 The tame Hecke algebra

For any $l \in \mathbb{N}$ coprime to Np, the Hecke operator T_l on modular forms can be defined as usual using Hecke correspondences. To spell this out, let us focus for simplicity on the case that l is a prime. Consider the perfectoid modular curve

$$\mathcal{X}_{\Gamma(p^\infty)\times\Gamma_0(l)} = \mathcal{X}_{\Gamma(p^\infty)} \times_{\mathcal{X}} \mathcal{X}_{\Gamma_0(l)} \sim \varprojlim_n \mathcal{X}_{\Gamma(p^n)\times\Gamma_0(l)}.$$

The maps $\pi_{1,2} : \mathcal{X}_{\Gamma_0(l)} \to \mathcal{X}$ given by $\pi_1 : (E, G) \mapsto E$ and $\pi_2 : (E, G) \mapsto E/G$, respectively, restrict to $\pi_{1,2} : \mathcal{X}_{\Gamma_0(l)}(\epsilon) \to \mathcal{X}(\epsilon)$ and induce Cartesian squares

$$\begin{tikzcd}
\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[d] & \mathcal{X}_{\Gamma(p^\infty)\times\Gamma_0(l)}(\epsilon)_a \arrow[l, "\pi_{2,\infty}"'] \arrow[r, "\pi_{1,\infty}"] \arrow[d] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[d] \\
\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a & \mathcal{X}_{\Gamma_0(p)\times\Gamma_0(l)}(\epsilon)_a \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] & \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a.
\end{tikzcd}$$

Lemma 2.5.1. *There is a natural isomorphism* $\pi_2^*\omega^\kappa_{\Gamma_0(p)} = \pi_1^*\omega^\kappa_{\Gamma_0(p)}$.

Proof. It will follow from Lemma 2.8.4 below that

$$\pi_1^*\omega^\kappa = \{f \in \mathcal{O}_{\mathcal{X}_{\Gamma(p^\infty)\times\Gamma_0(l)}(\epsilon)_a} \mid \gamma^* f = \pi_1^*\kappa^{-1}(c\mathfrak{z} + d)f \text{ for all } \gamma \in \Gamma_0(p)\}.$$

The same applies to $\pi_2^*\omega^\kappa$, so we are left to see that $\pi_1^*\kappa^{-1}(c\mathfrak{z} + d) = \pi_2^*\kappa^{-1}(c\mathfrak{z} + d)$. But this follows from the fact that $\pi_{\mathrm{HT}} \circ \pi_1 = \pi_{\mathrm{HT}} \circ \pi_2$, as can be seen by observing that the isogeny $E \to E/G$ induces an isomorphism of Hodge–Tate exact sequences. □

Since the morphism $\pi_1 : \mathcal{X}_{\Gamma_0(p)\times\Gamma_0(l)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$ is finite locally free, it gives rise to a trace map $\mathrm{Tr} : \pi_{1*}\mathcal{O}_{\mathcal{X}_{\Gamma_0(p)\times\Gamma_0(l)}(\epsilon)_a} \to \mathcal{O}_{\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a}$. This induces a map $\mathrm{Tr} : \pi_{1*}\pi_1^*\omega^\kappa \to \omega^\kappa$.

Definition 2.5.2. The Hecke operator T_l on $M_{\kappa,\Gamma_0(p)}(\epsilon)$ is defined as the composition

$$T_l : M_{\kappa,\Gamma_0(p)}(\epsilon) \xrightarrow{\pi_2} \Gamma(\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a, \pi_2^*\omega^\kappa_{\Gamma_0(p)})$$

$$\xrightarrow{\sim} \Gamma(\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a, \pi_1^*\omega^\kappa_{\Gamma_0(p)}) \xrightarrow{\kappa(l)l^{-1}\,\mathrm{Tr}} \Gamma(\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a, \omega^\kappa_{\Gamma_0(p)}) = M_{\kappa,\Gamma_0(p)}(\epsilon).$$

Via the isomorphism $\mathrm{AL} : M_{\kappa,\Gamma_0(p)}(p\epsilon) \xrightarrow{\sim} M_\kappa(\epsilon)$, after replacing ϵ by $p\epsilon$, this also defines an operator on $M_\kappa(\epsilon)$. Here the factor of $\kappa(l)l^{-1}$ follows the standard normalisation convention from e.g. [34], (1.11.0.2), which gives rise to the usual formula in q-expansions (1.11.1.2).

2.5.2 The U_p-operator

One defines the U_p-operator like in [3]. To see what this looks like in our setting, we note that via Atkin–Lehner isogenies, the usual Hecke correspondence for U_p translates into

$$\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \xleftarrow{\iota} \mathcal{X}_{\Gamma_0(p)}(p^{-1}\epsilon)_a \xrightarrow{\phi} \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$$

where ϕ is the Frobenius lift from (5). By Proposition 2.4.1, there is a canonical isomorphism

$$F : \iota^*\omega \xrightarrow{\sim} \phi^*\omega^\kappa.$$

Definition 2.5.3. The U_p-operator is defined to be the composition

$$U_p : M_\kappa(\epsilon) \xrightarrow{\iota} M_\kappa(p^{-1}\epsilon) = \Gamma(\mathcal{X}(p^{-1}\epsilon), \iota^*\omega^\kappa) \to$$

$$\xrightarrow{F^{-1}} \Gamma(\mathcal{X}(p^{-1}\epsilon), \phi^*\omega^\kappa) \xrightarrow{p^{-1}\,\mathrm{Tr}} \Gamma(\mathcal{X}(\epsilon), \omega^\kappa) = M_\kappa(\epsilon)$$

where $p^{-1}\,\mathrm{Tr} : \phi_*\phi^*\omega^\kappa \to \omega^\kappa$ is the normalised trace of the finite locally free morphism ϕ.

One easily checks that the operators thus defined have the desired effect on q-expansions, and in particular agree with the usual Hecke operators on classical subspaces.

Remark 2.5.4. For the perfectoid modular forms, one can define an action of the tame Hecke algebra exactly like in the finite level case. The definition of a U_p-operator at infinite level is more problematic: There is still an isomorphism $\iota^*\omega^{\kappa}_{\Gamma_0(p^\infty)} = \phi^*\omega^{\kappa}_{\Gamma_0(p^\infty)}$, which can be defined like in the proof of Proposition 2.4.1. However, since the isomorphism ϕ becomes an isomorphism $\mathcal{X}_{\Gamma_0(p^\infty)}(p^{-1}\epsilon) \xrightarrow{\sim} \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)$, its trace is simply the identity. In particular, a U_p-operator defined this way does not agree with the U_p-operator on the finite level subspaces. Alternatively, one could force the definition of a compatible operator by composing with the trace $M^{\mathrm{perf}}_{\kappa}(\epsilon) \to M_{\kappa}(\epsilon)$. But any eigenform of this operator would then of course be in $M_\kappa(\epsilon)$ already, so this action would not give any "new" eigenforms.

Definition 2.5.5. We define the integral Hecke algebra for κ to be the free commutative $\mathbb{Z}_p$-algebra generated by the T_l for primes l coprime to pN as well as a variable U_p°. We shall see that this algebra has a natural action on the space of integral modular forms $M^+_\kappa(\epsilon)$ when we let U_p° act as $T_\kappa^\delta \cdot U_p$ for $\delta = 3/p^{r_0+1}(p-1)$. The normalisation of the latter is chosen to obtain an operator that respects the integral subspace, in a way that is compatible in families extending to the boundary. For $|T_\kappa| \geq |p|$, the factor simplifies to $T_\kappa^\delta = p^{3\epsilon_\kappa/(p-1)}$.

2.6 Comparison with Chojecki–Hansen–Johansson's construction

Our construction of modular forms is of course just a minor modification of the construction in [17], and we do not claim any originality. We note that the article outlines the construction but only carries out all details in the quaternionic case: We refer to [10], §2 for some complements on how to deal with the boundary in the modular curve case. The difference between the construction in [17] and our present definition is that Chojecki–Hansen–Johansson use the canonical locus $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_c \to \mathcal{X}_{\Gamma_0(p)}(\epsilon)_c \xrightarrow{\sim} \mathcal{X}(\epsilon)$ rather than $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$. The two loci are canonically isomorphic via the automorphism of $\mathcal{X}_{\Gamma(p^\infty)}$ defined by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ whose action exchanges the two loci over the Atkin–Lehner isomorphism

$$\begin{tikzcd}
\mathcal{X}_{\Gamma(p^\infty)}(p\epsilon)_a \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)", "\sim"'] \arrow[d] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_c \arrow[d] \\
\mathcal{X}_{\Gamma_0(p)}(p\epsilon)_a & \mathcal{X}(\epsilon). \arrow[l, "\psi", "\sim"']
\end{tikzcd}$$

It identifies the $\Gamma_0(p)$-action on the right hand side with the $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$-conjugated action on the left. This leads to a twist in the automorphic factor, and recovers the definition in [17],

$$\omega^\kappa = \psi^*\omega^\kappa_{\Gamma_0(p)} \xrightarrow{\sim} \left\{ f \in q_*\mathcal{O}_{\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_c} \,\middle|\, \gamma^* f = \kappa^{-1}(b\mathfrak{z}+d) f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p) \right\}.$$

2.7 Comparison with Pilloni's construction

One can show using the canonical isomorphism $\pi^*_{\mathrm{HT}}\mathcal{O}(1) = q^*\omega$ of [50], Theorem IV.1.1.(v) and an explicit trivialisation of $\mathcal{O}(1)$ on $\mathbb{A}^1 \subseteq \mathbb{P}^1$ that in the case of κ of the form $x \mapsto x^k$ for $k \in \mathbb{N}$, the sheaf ω^κ is canonically isomorphic to the k-th tensor power $\omega^{\otimes k}$ of ω, as the notation suggests. In particular, in this case the above definition recovers Katz' definition of p-adic overconvergent modular forms of classical weight k. Elaborating on this argument, one can compare our definition of modular forms with the one due to Pilloni [46] and in the integral case due to Andreatta–Iovita–Pilloni, [3], as we shall now briefly discuss.

2.7.1 The Igusa tower in characteristic 0

We now recall the construction of the Pilloni-torsor in this setting: The original reference is [46], but we follow [10] since we wish to get a comparison of the integral sheaves.

Let $n \in \mathbb{N}$ and let $\epsilon < (p-1)/p^n$. Then the p^n-torsion $\mathcal{E}[p^n]$ of the semi-abelian scheme $\mathcal{E} \to \mathcal{X}(\epsilon)$ has a universal canonical subgroup $H_n \to \mathcal{X}(\epsilon)$ of rank p^n. Its dual $H_n^\vee$ is étale locally isomorphic to $\underline{\mathbb{Z}/p^n\mathbb{Z}}$. We can therefore form the étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor $\mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}(\epsilon)$ which relatively parametrises isomorphisms $\underline{\mathbb{Z}/p^n\mathbb{Z}} \to H_n^\vee$. We remark that over the canonical isomorphism $\psi_n : \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}(p^{-n}\epsilon)$, we have a canonical isomorphism $\mathcal{X}_{\Gamma_1(p^n)}(\epsilon)_a \xrightarrow{\sim} \mathcal{X}_{\mathrm{Ig}(p^n)}(p^{-n}\epsilon)$. We could therefore state all of the following also in terms of the anticanonical tower, but for the purpose of this section we find that it simplifies the exposition if we work from the point of Igusa level structures.

2.7.2 The Pilloni-torsor

We now construct the Pilloni-torsor in our setting, for which we work in the category of reduced topologically finite type (therefore stably uniform) analytic adic space over $(K, \mathcal{O}_K)$.

Fix $n \in \mathbb{N}$ and let $\epsilon < (p-1)/p^n$. Recall that $\omega = e^*\Omega_{\mathcal{E}|\mathcal{X}}$ denotes the sheaf of invariant differentials of the universal semi-abelian scheme $\mathcal{E} \to \mathcal{X}(\epsilon)$, where $e : \mathcal{X}(\epsilon) \to \mathcal{E}$ is the identity section. Let us denote by $\mathcal{T}(\epsilon) \to \mathcal{X}(\epsilon)$ the analytic total space of ω, this is the adic space representing the functor of sections of ω. We denote by $\mathcal{T}_n(\epsilon)$ the pullback to $\mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)$. Then by [46], Théorème 3.1 and Proposition 3.2, the Pilloni-torsor $\mathcal{F}_n(\epsilon) \to \mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)$ is the open subspace $\mathcal{F}_n(\epsilon) \subseteq \mathcal{T}_n(\epsilon)$ defined by the following functor of points:

Let (R, R°) be a stably uniform complete affinoid $(K, \mathcal{O}_K)$-algebra. Let $x : \mathrm{Spa}(R, R^\circ) \to \mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)$ be a point corresponding to a semi-abelian scheme $E|R$. Let E_0 be the unique extension to a semi-abelian scheme over R°, defined via $\mathcal{X}^*(R) = X(R^\circ)$. Then by the assumption on ϵ, the scheme E_0 admits a canonical subgroup $H_n(E_0) \subseteq E_0$ with generically étale dual $H_n(E_0)^\vee$. The point x defines a trivialisation $\alpha : \underline{\mathbb{Z}/p^n\mathbb{Z}}_R \to H_n^\vee$, by which we mean a morphism of group schemes over R° that becomes an isomorphism over R.

The closed immersion $H_n \hookrightarrow E_0$ induces a projection $\pi : \omega_E \to \omega_{H_n}$ of R°-modules. Together with the Hodge–Tate morphism, these maps fit into a commutative diagram

$$\begin{tikzcd} & & \omega_{E_0} \arrow[d, two heads, "\pi"] \\ \underline{\mathbb{Z}/p^n\mathbb{Z}} \arrow[r, "\alpha"] & H_n^\vee \arrow[r, "\mathrm{HT}"] & \omega_{H_n}. \end{tikzcd}$$

We now define $\mathcal{F}_n(\epsilon)_x \subseteq \mathcal{T}_n(\epsilon)_x = \omega_E$, the fibre of $\mathcal{F}_n(\epsilon)(R, R^\circ)$ over x, by

$$\mathcal{F}_n(\epsilon)_x := \{ w \in \mathcal{T}_n(\omega)_x = \omega_{E_0} \mid \pi(w) \in \mathrm{HT} \circ \alpha(1) \}.$$

Lemma 2.7.1. *1. The functor $\mathcal{F}_n(\epsilon)$ is represented by an open subspace $\mathcal{F}(\epsilon) \subseteq \mathcal{T}(\epsilon)$.*

2. *The natural map $f : \mathcal{F}_n(\epsilon) \to \mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)$ is a torsor in the analytic topology for the adic subgroup $B_n^1 := (1 + p^n \mathrm{Hdg}^{-p^n/(p-1)} \hat{\mathbb{G}}_a) \subseteq \hat{\mathbb{G}}_m \times \mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)$, where Hdg is the subsheaf of ideals of $\mathcal{O}_{\mathfrak{X}(\epsilon)}$ defined as the preimage of the sheaf of ideals $\mathrm{Ha} \cdot \omega^{\otimes(1-p)} \subseteq \mathcal{O}_{\mathfrak{X}(\epsilon)}/p$.*

3. *The natural map $g : \mathcal{F}_n(\epsilon) \to \mathcal{X}(\epsilon)$ is a torsor in the étale topology for $B_n := \mathbb{Z}_p^\times \cdot B_n^1$.*

4. *We have $(\mathcal{O}^+_{\mathcal{F}_n(\epsilon)})^{B_n} = \mathcal{O}^+_{\mathcal{X}(\epsilon)}$.*

Proof. The first three statements follow after base-change to K from [46], Théorème 3.1 and Proposition 3.2. For 5, one first sees in the analytic topology that $(f_* \mathcal{O}^+_{\mathcal{F}_n(\epsilon)})^{B_n^1} = \mathcal{O}^+_{\mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)}$. The statement then follows since $\mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}(\epsilon)$ is a finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor. $\square$

We fix the following setup, which allows one to define Pilloni's sheaf of modular forms:

Assumption 2.7.2. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight. Let $k = 0$ if $|p| \geq |T_\kappa|$ and otherwise let $k \in \mathbb{Z}_{\geq 0}$ be such that $|T_\kappa|^{p^{k+1}} \leq |p| \leq |T_\kappa|^{p^k}$. Let $n := k + r_0$. Let $0 \leq \epsilon \leq \epsilon_\kappa$.

One then sees as explained in [3], §5.3.1 that κ extends to an invertible function

$$\kappa^{\mathrm{an}} : B_n = \mathbb{Z}_p^\times(1 + p^n \mathrm{Hdg}^{-p^n/(p-1)}\hat{\mathbb{G}}_a) \to \hat{\mathbb{G}}_m.$$

Definition 2.7.3. The sheaf of modular forms on $\mathcal{X}(\epsilon)$ according to Pilloni is

$$\omega^\kappa_{P,n} := \mathcal{O}_{\mathcal{F}_n(\epsilon)}[\kappa^{-1}] = \{s \in f_*\mathcal{O}_{\mathcal{F}_n(\epsilon)} \mid \gamma^* s = \kappa^{-1}(\gamma)s \text{ for } \gamma \in B_n\}.$$

We also define $\omega^{\kappa,+}_{P,n}$ by using the $\mathcal{O}^+$-sheaf instead. Here the right hand side is shorthand for the following condition: Let $U \subseteq \mathcal{X}(\epsilon)$ be any open subset. Then $\omega^\kappa_{P,n}(U)$ consists of those $s \in f_*\mathcal{O}_{\mathcal{F}_n(\epsilon)}(U)$ that make the following diagram commutes:

$$\begin{tikzcd}
B_n|_U \times \mathcal{F}_n(\epsilon)|_U \arrow[r, "m"] \arrow[d, "\pi_2"] & \mathcal{F}_n(\epsilon)|_U \arrow[d, "s"] \\
\mathcal{F}_n(\epsilon)|_U \arrow[r, "s"] & \mathbb{A}^1
\end{tikzcd}$$

Proposition 2.7.4. *Let κ, ϵ, n be as in Assumption 2.7.2. Then $\omega^{\kappa,+}_{P,n}$ is an invertible $\mathcal{O}^+$-module on $\mathcal{X}(\epsilon)$. In particular, $\omega^\kappa_{P,n}$ is an invertible $\mathcal{O}_{\mathcal{X}(\epsilon)}$-module.*

Proof. Since the original setup in [46] does not consider integral structures, we follow [3]: We argue by base-change from the relative situation. By Lemma 2.7.1.4, we have $(\mathcal{O}^+_{\mathcal{F}_n(\epsilon)})^{B_n} = \mathcal{O}^+_{\mathcal{X}(\epsilon)}$ and it therefore suffices to show that $\omega^{\kappa,+}_{P,n}$ locally admits an invertible section.

Assume first that $|p| \leq |T_\kappa|$. Andreatta–Iovita–Pilloni prove in [3], Théorème 5.1.1 the following: Let $I = [p^k, p^{k+1}]$. Over the formal model $\mathfrak{W}_I := \mathrm{Spf}(\mathbb{Z}_p[[T]]\langle T^{p^{k+1}}/p, p/T^k\rangle)$ of the open subspace $\mathcal{W}_I := \mathcal{W}(|T|^{p^{k+1}} \leq |p| \leq |T^{p^k}|)$ of weight space, there is a relative formal modular curve $\mathfrak{X}_{r,I}$ defined by the condition $|\mathrm{Ha}|^{p^{r+1}} \geq |T|$ and a relative Pilloni-torsor $\mathfrak{F}_{n,r,I} \to \mathfrak{X}_{r,I}$. One can define a sheaf $\mathfrak{w}_{n,r,I}$ on the formal scheme $\mathfrak{X}_{r,I}$ precisely as above. The base-change along the point $\mathrm{Spf}(\mathcal{O}_K) \to \mathfrak{W}_I$ corresponding to κ is defined by sending $T \mapsto T_\kappa$. It therefore sends the condition $|\mathrm{Ha}|^{p^{r_0+1}} \geq |T|$ to $|\mathrm{Ha}|^{p^{r_0+1}} \geq |T_\kappa|$. Since by our assumption, we have $|p|^\epsilon \geq |T_\kappa|^{1/p^{r_0+1}}$, we thus obtain a morphism of locally ringed spaces $s : (\mathcal{X}(\epsilon), \mathcal{O}^+) \to \mathfrak{X}_{r,I}$. It is then clear from the definition that we have $\omega^{\kappa,+}_{P,n} = s^*\mathfrak{w}_{n,r,I}$. In particular, the former is locally free of rank 1 since the latter is.

The proof in the case $|p| \geq |T_\kappa|$ is analogous, using the interval $I = [0, 1]$ instead. $\square$

2.7.3 Comparison

The comparison of our construction to Pilloni's relies on the fact that ω becomes trivial after pull-back to $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$, and we can give an explicit section using π_{HT}. In the following, we essentially follow [17] and [10], although our comparison will be a bit different: For instance, we note that the comparison in [17] is local, whereas our definition will be global on $\mathcal{X}(\epsilon)$.

We start by trivialising $\mathcal{O}(1)$ locally on $\mathbb{P}^1$ via an explicit non-vanishing section, and pulling this back along the Hodge-Tate period map. This section can be defined as follows: Regard $\mathbb{P}^1(K)$ as the moduli space of lines $L \subseteq K^2$, or equivalently of one-dimensional quotients $K^2 \twoheadrightarrow Q := K^2/L$, then we can regard the total space of the line bundle $\mathcal{O}(1)$ as the moduli space of points of Q. The image of $(1, 0) \in K^2 \to Q$ defines such a point, and thus gives a global section s of $\mathcal{O}(1)$. Away from $(1 : 0) = \infty \in \mathbb{P}^1$, this section is non-vanishing. Therefore s gives a trivialisation of $\mathcal{O}(1)$ over the chart $\mathbb{A}^1 \subseteq \mathbb{P}^1$ away from ∞ with parameter $(z : 1)$. One checks that the action of $\Gamma_0(p)$ on this chart is described by

$$\gamma^* s = (cz + d)s \quad \text{for all } \gamma \in \Gamma_0(p). \tag{6}$$

Let now $\mathfrak{s} := \pi_{\mathrm{HT}}^* s$. Then we see from the moduli description of s that for any (C, C^+)-point $x \in \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ corresponding to an elliptic curve E with trivialisation $\alpha : \mathbb{Z}_p^2 \to T_pE$, the point $\mathfrak{s}(x) \in \omega_E$ is given by the image of $\alpha(1,0)$ under $\mathrm{HT} : T_pE \to \omega_E$.

By the moduli description of $\mathcal{T}(\epsilon)$, we may regard $\mathfrak{s}$ as a morphism of adic spaces

$$\mathfrak{s} : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{T}(\epsilon).$$

Recall that we have defined $\mathfrak{z} = \pi_{\mathrm{HT}}^* z$. It then follows from equation (6) that we have

$$\gamma^* \mathfrak{s} = (c\mathfrak{z} + d)\mathfrak{s} \quad \text{for all } \gamma \in \Gamma_0(p). \tag{7}$$

Lemma 2.7.5. *The map $\mathfrak{s}$ factors through the Pilloni-torsor and defines a map*

$$\mathfrak{s} : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{F}_n(\epsilon).$$

Proof. There is a natural map $\varphi : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\mathrm{Ig}(p^n)}(\epsilon)$ defined by sending a point valued in some stably uniform adic ring (R, R°) corresponding to a an elliptic curve E and a trivialisation $\alpha : \mathbb{Z}_p^2 \to T_pE$ to the trivialisation of $H_n^\vee(E)$ given by the composition

$$\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{(1,0)} \mathbb{Z}_p^2/p^n \xrightarrow{\alpha \bmod p^n} E[p^n] \to E[p^n]/H_n(E) = H_n^\vee(E).$$

By functoriality of the Hodge-Tate map we then have a commutative diagram

$$\begin{tikzcd} T_pE \arrow[r, "\mathrm{HT}"] \arrow[d] & \omega_E \arrow[d, "\pi"] \\ H_n^\vee = E[p^n]/H_n \arrow[r, "\mathrm{HT}"] & \omega_{H_n}. \end{tikzcd}$$

In the case of (R, R°) an algebraically closed complete field, we then have $\mathfrak{s}(x) = \mathrm{HT} \circ \alpha(1,0)$ by the moduli description of $\mathfrak{s}$, and we conclude from the description of φ that this defines an element of $\mathcal{F}_n(\epsilon)(R, R^\circ)$ as desired. Since it suffices to check on points that the image of $\mathfrak{s}$ is contained in the subspace $\mathcal{F}_n(\epsilon) \subseteq \mathcal{T}(\epsilon)$, this proves the Lemma. $\square$

Combining equation (7) with Lemmas 2.2.2.1 and 2.7.5, we get a commutative diagram

$$\begin{tikzcd} \Gamma_0(p) \times \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r, "m"] \arrow[d, "(c\mathfrak{z}+d)\times\mathfrak{s}"'] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r] \arrow[d, "\mathfrak{s}"'] & \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \arrow[d, "q"] \\ B_n \times \mathcal{F}_n(\epsilon) \arrow[r, "m"] & \mathcal{F}_n(\epsilon) \arrow[r] & \mathcal{X}(\epsilon) \end{tikzcd} \tag{8}$$

where the horizontal maps on the left are the respective actions and q is the forgetful map. Combining this with the isomorphism $u_p : \mathcal{X}_{\Gamma(p^\infty)}(p\epsilon)_a \to \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ and the calculation surrounding diagram (11), we also obtain a commutative diagram

$$\begin{tikzcd} \Gamma_0(p) \times \mathcal{X}_{\Gamma(p^\infty)}(p\epsilon)_a \arrow[r, "m"] \arrow[d, "(c\mathfrak{z}+d)\times u_p^*\mathfrak{s}"'] & \mathcal{X}_{\Gamma(p^\infty)}(p\epsilon)_a \arrow[r] \arrow[d, "u_p^*\mathfrak{s}"'] & \mathcal{X}_{\Gamma_0(p)}(p\epsilon)_a \arrow[d, "\psi^{-1}"', "\wr"] \\ B_n \times \mathcal{F}_n(\epsilon) \arrow[r, "m"] & \mathcal{F}_n(\epsilon) \arrow[r] & \mathcal{X}(\epsilon) \end{tikzcd} \tag{9}$$

Proposition 2.7.6. *Pullback along $u_p^*\mathfrak{s}$ defines natural isomorphisms of sheaves on $\mathcal{X}(\epsilon)$*

$$\omega^{+,\kappa} = \omega_{P,n}^{+,\kappa} \quad \textit{and} \quad \omega^\kappa = \omega_{P,n}^\kappa.$$

In particular, for $\epsilon \leq \epsilon_\kappa$, the sheaf $\omega^{+,\kappa}$ is an invertible $\mathcal{O}^+$-module and ω^κ is an invertible $\mathcal{O}$-module on $\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$. They are trivial on any open $U \subseteq \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a$ where ω is trivial.

The proof is not hard, but since we will need the same argument several times again, we first abstract from diagram (9) and formalise the situation in the following subsection.

2.8 Context: Invertible modules from torsors and cocycles

In order to formalise the situation of diagram (9), we briefly discuss the technical framework underlying the definitions of modular forms. We first consider the following simpler setting:

Remark 2.8.1. Let G be a finite group and let $\pi : Y \to X$ be an étale Galois cover of schemes with group G. The Cartan–Leray spectral sequence for the sheaf $\mathcal{O}^\times$ on $X_{\text{ét}}$ gives rise to a map

$$H^1(G, \mathcal{O}(Y)^\times) \to H^1_{\text{ét}}(X, \mathcal{O}^\times) = H^1_{\text{Zar}}(X, \mathcal{O}^\times)$$

which we may interpret as sending any 1-cocycle $\kappa : G \to \mathcal{O}(Y)^\times$ in group cohomology to an invertible $\mathcal{O}_X$-module $\mathcal{F}_\kappa$. This map can be made explicit as follows: The cocyle κ can be used to define a G-action on Y by sending $\gamma \cdot f \mapsto \gamma^* f \cdot \kappa(f)$. Then the sheaf $\mathcal{F}$ is the subsheaf of $\pi_*\mathcal{O}_Y$ of invariants with respect to this twisted action:

$$\mathcal{F}_\kappa = \pi_*\mathcal{O}_Y[\kappa^{-1}] = \{f \in \pi_*\mathcal{O}_Y | \gamma^* f = \kappa^{-1}(\gamma) f\}. \tag{10}$$

Another perspective on the same construction would be that κ gives rise to a descent datum with respect to $\pi : Y \to X$ for a line bundle, which in this case is effective.

The definitions of modular forms from the last section are motivated by the construction in the remark. However, there are various obstacles that keep us from a straightforward adaptation: For instance, we are working in the pro-étale site, where descent is not necessarily effective (cf [17], paragraph before §1.2, and [38], Example 8.1.6). Also, we will want to apply this construction to morphisms which are, for example, formal models of torsors but not actually torsors themselves. Nevertheless, one can still copy equation (10), and the task is then to prove that this is really a line bundle, which takes two steps: proving that $(\pi_*\mathcal{O}_Y)^G = \mathcal{O}_X$, and locally constructing an invertible section. We now make this precise:

Let us work in a category $\mathcal{C}$ equipped with a final object and a functor $F : \mathcal{C} \to \mathrm{RS}$ to the category of ringed spaces. In our application, $\mathcal{C}$ will be the category of formal schemes, or adic spaces, over some fixed base, and F could be either the functor $X \mapsto (X, \mathcal{O}_X)$ or $X \mapsto (X, \mathcal{O}^+_X)$. For $X \in \mathcal{C}$, we write $\mathcal{O}_X := \mathcal{O}_{F(X)}$. Assume that $\mathcal{C}$ has objects $\mathbb{A}$ and $\mathbb{G}_m$ that represent the functors $X \mapsto \mathcal{O}(F(X))$ and $X \mapsto \mathcal{O}(F(X))^\times$, respectively, as well as an action $m : \mathbb{G}_m \times \mathbb{A} \to \mathbb{A}$. Here and in the following, whenever we write down a fibre product in $\mathcal{C}$, we will tacitly assume that this exists. Finally, we assume that for $X \in \mathcal{C}$ and any open $U \subseteq F(X)$, we can associate to U an object $X|_U$ with an arrow $i : X|_U \to X$ in $\mathcal{C}$ in a functorial way such that $F(i) = (U \to X)$, and that arbitrary fibre products with i exist.

Definition 2.8.2. Let $X, Y \in \mathcal{C}$. Let G be an X-group, i.e. a group object in $\mathcal{C}/X$.

1. Let $\pi : X_\infty \to X$ be a morphism in $\mathcal{C}$. We say that G acts on π if we have a left group action of G on $\pi : X_\infty \to X$ in the category $\mathcal{C}/X$, i.e. a morphism $m_G : G \times_X X_\infty \to X_\infty$ that satisfies the usual condition in terms of commutative diagrams.

2. If H is a Y-group and $h : X \to Y$ is a morphism, then a morphism $c : G \times_X X_\infty \to H$ over h is a 1-cocycle if it satisfies (to be interpreted in terms of a suitable diagram):

$$c(\gamma_1 \cdot \gamma_2, x) = c(\gamma_1, \gamma_2 \cdot x) \cdot c(\gamma_2, x) \text{ for all } \gamma_1, \gamma_2, x \in G \times_X G \times_X X_\infty,$$

The relation to group cohomology is that in the case that $\mathcal{C}$ is the category of adic spaces and $H = \mathbb{G}_m$ and $G = \underline{G_0}$, the map c corresponds to a function $c \in \mathrm{Map}_{\mathrm{cts}}(G_0, \mathcal{O}_{X_\infty}(X_\infty)^\times)$ which is a 1-cocycle in group cohomology for the right action of G_0 on $\mathcal{O}_{X_\infty}(X_\infty)^\times$. An example we have in mind is $c\mathfrak{z} + d : \underline{\Gamma_0(p)} \times \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \hat{\mathbb{G}}_m$ on $X = Y = \mathrm{Spa}(K)$.

Definition 2.8.3. Let $\kappa : G \times_X X_\infty \to \mathbb{G}_m$ be a 1-cocycle. We define a sheaf on X by

$$\mathcal{O}_{X_\infty}[\kappa] = \{s \in \pi_*\mathcal{O}_{X_\infty} \mid s(gx) = \kappa(g,x)s(x) \text{ for all } g, x \in G \times_X X_\infty\}.$$

Explicitly, for any open $U \subseteq X$, the sections $\mathcal{O}_{X_\infty}[\kappa](U)$ consist of those $s : X_{\infty|U} \to \mathbb{A}$ for which $(G \times_X X_\infty)_{|U} \xrightarrow{m} X_{\infty|U} \xrightarrow{s} \mathbb{A}$ coincides with $(G \times_X X_\infty)_{|U} \xrightarrow{\kappa\times s} \mathbb{G}_m \times \mathbb{A} \xrightarrow{m} \mathbb{A}$.

Since G leaves π invariant, π induces a map $\mathcal{O}_X \to (\pi_*\mathcal{O}_{X_\infty})^G$, and thus $\mathcal{O}_{X_\infty}[\kappa]$ is always an $\mathcal{O}_X$-module. If this map is even an isomorphism, then $\mathcal{O}_{X_\infty}[\kappa]$ is a line bundle if it locally admits a section that is invertible as a function on $\mathcal{O}_{X_\infty}$. In particular:

Lemma 2.8.4. *Let X, Y be objects in $\mathcal{C}$. Let G be an X-group that acts on a morphism $f : X_\infty \to X$. Let H be a Y-group that acts on a morphism $g : Y_\infty \to Y$. Let $\phi : G \times_X X_\infty \to H$ be a 1-cocycle over a morphism $h : X \to Y$ that fits into a commutative diagram*

$$\begin{tikzcd}
G \times_X X_\infty \arrow[r, "m_G"] \arrow[d, "\phi\times h_\infty"] & X_\infty \arrow[r, "f"] \arrow[d, "h_\infty"] & X \arrow[d, "h"] \\
H \times_Y Y_\infty \arrow[r, "m_H"] & Y_\infty \arrow[r, "g"] & Y.
\end{tikzcd}$$

Let $\kappa : H \times_Y Y_\infty \to \mathbb{G}_m$ be a 1-cocycle. Suppose that $\mathcal{O}_{Y_\infty}[\kappa]$ is an invertible $\mathcal{O}_Y$-module that locally admits a section which is invertible as a function on Y_∞, and that $(f_*\mathcal{O}_{X_\infty})^G = \mathcal{O}_X$. Set $\kappa' := \kappa \circ (\phi, h_\infty) : G \times_X X_\infty \to \mathbb{G}_m$. Then κ' is a 1-cocycle, and the natural map

$$h^*(\mathcal{O}_{Y_\infty}[\kappa]) \to \mathcal{O}_{X_\infty}[\kappa']$$

is an isomorphism of $\mathcal{O}_X$-modules. In particular, $\mathcal{O}_{X_\infty}[\kappa']$ is an invertible $\mathcal{O}_X$-module.

Proof. Since $(f_*\mathcal{O}_{X_\infty})^G = \mathcal{O}_X$, it suffices to show that for any open $U \subseteq Y$ on which there is an invertible section $s \in \mathcal{O}_{Y_\infty}[\kappa]$, we have $h_\infty^* s \in \mathcal{O}_{X_\infty}[\kappa'](h^{-1}(U))$. But by the diagram, $s(h_\infty(\gamma \cdot x)) = s(\phi(\gamma, x) \cdot h_\infty(x)) = \kappa(\phi(\gamma, x), h_\infty(x)) \cdot s(h_\infty(x)) = \kappa'(\gamma, x) \cdot s(h_\infty(x))$. $\square$

proof of Proposition 2.7.6. We apply Lemma 2.8.4 with $\mathcal{C}$ the category of adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, and the functor F to locally ringed spaces picking out the $\mathcal{O}^+$-sheaf. We use diagram (9). For κ we use $B_n \times \mathcal{F}(\epsilon) \xrightarrow{\pi_1} B_n \xrightarrow{\kappa^{\mathrm{an}}} \hat{\mathbb{G}}_m$. By Lemma 2.1.3, the invariants condition is satisfied. We may thus apply Lemma 2.8.4, which gives an isomorphism $(\psi^{-1})^*\omega^{\kappa,+}_{P,n} = \omega^{\kappa,+}_{\Gamma_0(p)}$ on $\mathcal{X}_{\Gamma_0(p)}(p\epsilon)_a$. Applying ψ^* then shows $\omega^{\kappa,+}_{P,n} = \psi^*\omega^{\kappa,+}_{\Gamma_0(p)} = \omega^{\kappa,+}$. $\square$

2.9 Further alternative definitions

Throughout this section we fix a weight $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$, and let $0 \leq \epsilon \leq \epsilon_\kappa$ with $\epsilon \in \log|K|$.

Now that we know that the sheaves ω^κ and ω^κ are line bundles, we can verify that for any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the sheaves $\omega_{\Gamma_0(p^n)}$ are invertible modules as well, and can alternatively be described by pullback. This also gives an alternative definition of ω^{perf}.

Corollary 2.9.1. *Let $n \leq m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let $\lambda : \mathcal{X}_{\Gamma_0(p^m)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ be the natural map. Then the inclusion $\omega^{+,\kappa}_{\Gamma_0(p^n)} \subseteq \lambda_*\omega^{+,\kappa}_{\Gamma_0(p^m)}$ induces an isomorphism*

$$\omega^{+,\kappa}_{\Gamma_0(p^m)} = \lambda^*\omega^{+,\kappa}_{\Gamma_0(p^n)}.$$

In particular, these sheaves are all invertible $\mathcal{O}^+$-modules. Similarly for $\omega^\kappa_{\Gamma_0(p^n)}$.

Proof. Let $i : \Gamma_0(p^m) \hookrightarrow \Gamma_0(p^n)$ be the inclusion. We apply Lemma 2.8.4 to the diagram

$$\begin{tikzcd}
\Gamma_0(p^m) \times \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r, "m"] \arrow[d, "i\times\mathrm{id}"'] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r] \arrow[d, equal] & \mathcal{X}_{\Gamma_0(p^m)}(\epsilon)_a \arrow[d, "\lambda"'] \\
\Gamma_0(p^n) \times \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r, "m"] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[r] & \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a
\end{tikzcd}$$

where m is the group action. Here the invariants condition holds by Lemma 2.1.3.1. $\square$

Proposition 2.9.2. *Using the $\mathbb{Z}_p^\times$-torsor $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$, we have*

$$\omega^{\kappa,+,\mathrm{perf}} = \mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}[\kappa^{-1}] = \{f \in \mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a} \mid \gamma^* f = \kappa^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times\}.$$

Proof. The $\Gamma_0(p^\infty)$-equivariance means that for any $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ we have $\gamma^* f = \kappa^{-1}(d) f$. This is equivalent to $\mathbb{Z}_p^\times$-equivariance in d and invariance under matrices of the form $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, i.e. $\gamma \in \Gamma_1(p^\infty)$. But by Lemma 2.1.3, we have $(\mathcal{O}^+_{\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a})^{\Gamma_1(p^\infty)} = \mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}$. $\square$

In particular, via the inclusions $\omega^\kappa_{\Gamma_0(p^n)} \subseteq \omega^\kappa_{\Gamma_0(p^\infty)}$ for all n, this means that we may regard every modular form $f \in M_{\kappa,\Gamma_0(p^n)}(\epsilon)$ as a function on $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$. From this perspective, our definition of p-adic modular forms via modular curves could be interpreted as follows: Since for $n < \infty$ the map $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ is not a torsor, even away from the boundary, we have to go up to $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ to be able to say which perfectoid modular forms are of level $\Gamma_0(p^n)$: The latter is a $\Gamma_0(p^n)$-torsor away from the cusps, and we can use this to descend from $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ to $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$.

Proof of Proposition 2.4.1. The action of $u_{p^n} := \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[d] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[l, "u_{p^n}"'] \arrow[r, equal] \arrow[d] & \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \arrow[d] \\
\mathcal{X}_{\Gamma_0(p)}(\epsilon)_a & \mathcal{X}_{\Gamma(p^{n+1})}(\epsilon)_a \arrow[l, "v^n"'] \arrow[r, "\lambda"] & \mathcal{X}_{\Gamma_0(p)}(\epsilon)_a \\
 & \mathcal{X}_{\Gamma_0(p)}(p^{-n}\epsilon)_a . \arrow[ul, hook, "\iota^n"] \arrow[u, "\psi^n", "\sim"'] \arrow[ur, "\phi^n"'] &
\end{tikzcd} \tag{11}$$

where we recall that λ is the forgetful map and v^n may be defined simply by the bottom left triangle, and then sends $(E, D_{n+1}) \mapsto (E/D_n, D_{n+1}/D_n)$. In particular, the left side of the diagram shows that u_{p^n} in fact sends $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ into $\mathcal{X}_{\Gamma(p^\infty)}(p^{-n}\epsilon)_a$.

Arguing like in Corollary 2.9.1, we want to apply Lemma 2.8.4 to the left hand side of the diagram. For this we need to calculate the effect of u_{p^n} on the cocycle: We first note that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$u_{p^n}\gamma = (u_{p^n}\gamma u_{p^n}^{-1})u_{p^n} = \begin{pmatrix} a & p^n b \\ p^{-n}c & d \end{pmatrix} u_{p^n}$$

and thus $u_{p^n} : \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$ is equivariant with respect to the conjugation map $j : \Gamma_0(p^{n+1}) \to \Gamma_0(p)$ defined by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \begin{pmatrix} a & p^n b \\ p^{-n}c & d \end{pmatrix}$.

Next, we note that on $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, the action of u_{p^n} is given by $z \mapsto p^n z$. By $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariance of π_{HT}, this shows that $u_{p^n}^*\mathfrak{z} = p^n\mathfrak{z}$. Consider now the cocycle $\kappa^{-1}(\mathfrak{z},\gamma) := \kappa^{-1}(c\mathfrak{z}+d)$, then we conclude that for any $\gamma \in \Gamma_0(p^n)$,

$$\kappa^{-1} \circ (u_{p^n} \times j)(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \mathfrak{z}) = \kappa^{-1}(\begin{pmatrix} a & p^n b \\ p^{-n}c & d \end{pmatrix}, p^n\mathfrak{z}) = \kappa^{-1}(c\mathfrak{z}+d) = \kappa^{-1}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \mathfrak{z}).$$

Consequently, Lemma 2.8.4 shows that the left square in diagram (11) gives an isomorphism

$$v^{n*}\omega^+_{\Gamma_0(p)} = \omega^+_{\Gamma_0(p^{n+1})} = \lambda^{n*}\omega^+_{\Gamma_0(p)}.$$

Since $v^n \circ \psi^n = \iota^n$ and $\phi^n = \lambda \circ \psi^n$, applying ψ^{n*} then shows that

$$\iota^{n*}\omega^+_{\Gamma_0(p)} = \psi^{n*}\omega^+_{\Gamma_0(p^{n+1})} = \phi^{n*}\omega^+_{\Gamma_0(p)}.$$

Applying ψ^* once more now gives the desired statement for $\omega^+ = \psi^*\omega^+_{\Gamma_0(p)}$.

The statement about q-expansions follows from [27], Proposition 2.10. $\square$

3 t-adic and perfectoid modular forms

In this section, we recall Andreatta–Iovita–Pilloni's [3] construction of t-adic modular forms: Compared to the last section, this means we replace $\mathbb{Z}_p$ with $\mathbb{F}_p[[t]]$ and discuss non-archimedean modular curves and modular forms over the field $\mathbb{F}_p((t))$ and its perfectoid extensions. Almost all of the material in this section is due to Andreatta–Iovita–Pilloni, but we shall discuss a few complements and adapt the construction to our setting: For this, we work in parallel over both $\mathbb{F}_p((t))$ as well as a perfectoid field extension K of $\mathbb{F}_p((t^{1/p^\infty}))$. Note that any perfectoid field extension of $\mathbb{F}_p((t))$ in characteristic p contains an isomorphic copy of $\mathbb{F}_p((t^{1/p^\infty}))$, so the structure map basically just amounts to fixing a uniformiser t. Second, our setup allows us to use perfectoid spaces to define perfectoid t-adic modular forms – for the purpose of this section, this will just be a minor technical modification. It is only in section §5 that we make use more systematically of the perfectoid structure. In particular, we leave it until then to upgrade the conceptual analogies between the p-adic and t-adic modular curves to a precise comparison of both settings via tilting isomorphisms.

There are two main reasons why who choose to work over both $\mathbb{F}_p((t))$ and K instead of just sticking to the latter, which is the one we use later: The first is that we want to compare the perfectoid setting to the one of Andreatta–Iovita–Pilloni. The second is that in several places we need the relation to the Noetherian case for technical reasons, for example since it is not clear that normalisations are well-defined for non-excellent formal schemes.

3.1 The setting of Andreatta–Iovita–Pilloni: t-adic modular curves

Let K be a non-archimedean field extension of $\mathbb{F}_p((t))$ with ring of integers R. Like in the last section, we denote by X_R the compactified modular curve over R of tame level N coprime to p, and by X_K its generic fibre. These are just the base changes of $X_{\mathbb{F}_p}$ to R and K. We denote by $\mathfrak{X}_R$ the t-adic completion of X_R. Let $\mathcal{X}_K$ be the t-adic analytification over $\mathrm{Spa}(K,R)$ of X_K. Let $E_R \to X_R$ be the universal semi-abelian scheme with identity section e and conormal sheaf $\omega := e^*\Omega^1_{E_R|X_R}$ and let $\mathcal{E}_K \to \mathcal{X}_K$ be its analytification. Since we are in characteristic p, we have the Hasse invariant $\mathrm{Ha} \in \omega^{\otimes(p-1)}$.

Let $X_{R,\mathrm{ord}}$ be the ordinary locus. This is the affine open subscheme defined by the condition $\mathrm{Ha} \neq 0$. Let $\mathfrak{X}_{R,\mathrm{ord}}$ be its t-adic completion, this is the affine open subspace $\mathfrak{X}_{R,\mathrm{ord}} \subseteq \mathfrak{X}_R$ where Ha is invertible. Let $\mathcal{X}_{K,\mathrm{ord}} \subseteq \mathcal{X}_K$ be the open subspace obtained by analytification of $X_{K,\mathrm{ord}}$. This is the non-quasicompact, hence non-affinoid, open subspace obtained by removing the finitely many closed points where the Hasse invariant vanishes.

For any $0 \leq \epsilon \in \mathbb{Q}\cdot\log|K|$, we denote by $\mathcal{X}_K(\epsilon) \subseteq \mathcal{X}_K$ the affinoid open subspace

$$\mathcal{X}_K(\epsilon) := \mathcal{X}_K(|\mathrm{Ha}| \geq |t|^\epsilon).$$

Since $\mathrm{Ha} \neq 0$ on $\mathcal{X}_K(\epsilon)$ we have $\mathcal{X}_K(\epsilon) \subseteq \mathcal{X}_{K,\mathrm{ord}}$. On the other hand, we have inclusions

$$(\mathfrak{X}_{R,\mathrm{ord}})^{\mathrm{ad}}_\eta = \mathcal{X}_K(0) \subseteq \mathcal{X}_K(\epsilon) \subseteq \mathcal{X}_{K,\mathrm{ord}},$$

that is $\mathcal{X}_K(\epsilon)$ contains the locus $\mathcal{X}_K(0)$ of *ordinary reduction* where $|\mathrm{Ha}| \geq 1$. In order to distinguish "the two ordinary loci", we refer to them as the *generically ordinary locus* and the *locus of ordinary reduction*, respectively.

3.1.1 Igusa level structures

We recall from [36], Definition 12.3.1, that for elliptic curves in characteristic p, there is the moduli problem of Igusa structures:

Definition 3.1.1. Let S be any $\mathbb{F}_p$-algebra and let $E|S$ be an elliptic curve. For any $n \in \mathbb{N}$, an Igusa structure of level p^n for E is the data of a morphism

$$\alpha : \underline{\mathbb{Z}/p^n\mathbb{Z}}_S \to \ker(V^n : E^{(p^n)} \to E) \subseteq E^{(p^n)}[p^n]$$

that is a generator in the sense of Drinfeld level structures. If the elliptic curve is ordinary, this is equivalent to saying that α is an isomorphism of S-group schemes. In this case, we moreover have $\ker V^n = C_n^\vee$ where $C_n = \ker F^n \subseteq E[p^n]$ is the canonical subgroup of E, so we may equivalently regard the Igusa moduli problem as trivialising the dual of the canonical subgroup of E. This interpretation has the advantage to extend to the cusps, where we can still make sense of the canonical subgroup of the universal semi-abelian scheme. We let

$$X_{S,\mathrm{Ig}(p^n)} \to X_S$$

be the moduli scheme which relatively represents the Igusa moduli problem

Returning to our setting where K is an extension of $\mathbb{F}_p((t))$ with ring of integers R, let $\mathfrak{X}_{R,\mathrm{Ig}(p^n)} \to \mathfrak{X}_R$ be the completion of $X_{R,\mathrm{Ig}(p^n)} \to X_R$, and let $\mathcal{X}_{K,\mathrm{Ig}(p^n)} \to \mathcal{X}_K$ be the analytification of $X_{K,\mathrm{Ig}(p^n)} \to X_K$. These are all finite flat morphisms in their respective categories, and even étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsors over the (generically) ordinary locus. In particular, on analytifications the Igusa scheme restricts to a finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor

$$\mathcal{X}_{K,\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}_K(\epsilon).$$

It is clear from the moduli problem that for any $0 \leq m \leq n$ this factors through forgetful maps $\mathcal{X}_{K,\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}_{K,\mathrm{Ig}(p^m)}(\epsilon)$, so that we get an overconvergent analytic "Igusa tower"

$$\cdots \to \mathcal{X}_{K,\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}_{K,\mathrm{Ig}(p^{n-1})}(\epsilon) \to \cdots \to \mathcal{X}_K(\epsilon).$$

3.1.2 Formal models and the limit of the Igusa tower

Following [3], §4, let us for a moment restrict to the case of $K = \mathbb{F}_p((t))$ and $\epsilon = 1/p^{r+1}$. We also allow $\epsilon = 1$, i.e. $r = -1$: Strictly speaking, this case is not contained in [3], for the good reason that in mixed characteristic, $\epsilon = 1$ is too big for a canonical subgroup to exist. However, over $\mathbb{F}_p$ this is no problem, so we may include $\epsilon = 1$ since we shall later need it.

We wish to lighten notation and drop the index $\mathbb{F}_p((t))$ from notation. Since we still need to tell the modular curves over $\mathbb{F}_p((t))$ apart from their analogues over perfectoid fields, as well as from their characteristic 0 counterparts, we employ the following notation, which has the advantage to also be consistent with the notation of [3]:

Notation 3.1.2. We set $\mathcal{X}_r := \mathcal{X}_{\mathbb{F}_p((t))}(1/p^{r+1})$ and $\mathcal{X}_{r,\mathrm{Ig}(p^n)} := \mathcal{X}_{\mathbb{F}_p((t)),\mathrm{Ig}(p^n)}(1/p^{r+1})$.

The analytic adic space $\mathcal{X}_r$ has a canonical formal model $\mathfrak{X}_r \to \mathfrak{X}_{\mathbb{F}_p[[t]]}$ defined as follows: Locally over an affine open $U = \mathrm{Spf}(S) \subseteq \mathfrak{X}_{\mathbb{F}_p[[t]]}$ where ω becomes trivial, choose a trivialisation so that we can write $\mathrm{Ha} \in R$, unique up to a unit. We then set

$$\mathfrak{X}_r|_U := \mathrm{Spf}(S\langle X\rangle/(X\mathrm{Ha}^{p^{r+1}} - t)). \tag{12}$$

This is independent of the trivialisation up to canonical isomorphism, and therefore glues to a flat formal scheme $\mathfrak{X}_r \to \mathfrak{X}_{\mathbb{F}_p[[t]]}$, which is clearly a formal model of $\mathcal{X}_r$.

Lemma 3.1.3 ([3], Lemme 3.4)**.** *The formal scheme $\mathfrak{X}_r$ is normal and excellent.*

Proof. The case $r = -1$ is not stated in Lemme 3.4, but the proof by Serre's criterion goes through verbatim: $A = S\langle X\rangle/(X\mathrm{Ha} - t)$ is visibly a locally complete intersection, thus satisfies S_2. To verify R_1, let $\mathfrak{p}$ be a prime ideal of height 1. Either $t \notin \mathfrak{p}$ in which case $\mathfrak{p}$ corresponds to a point of a smooth rigid space and thus $A_{\mathfrak{p}}$ is regular. Or $t \in \mathfrak{p}$, then $A/t = S[X]/(X\mathrm{Ha})$. So $\mathfrak{p} = (\mathrm{Ha})$ or $\mathfrak{p} = (X)$, i.e. $\mathfrak{p}$ is principal and $A_{\mathfrak{p}}$ is regular.

As usual, excellence follows from the main result of [57]. $\square$

Due to these desirable properties, one can now obtain a formal model of the Igusa tower by defining $\mathfrak{X}_{r,\mathrm{Ig}(p^n)} \to \mathfrak{X}_r$ to be the normalisation of $\mathfrak{X}_r$ in $\mathcal{X}_{K,\mathrm{Ig}(p^n)}(\epsilon)$. This is well-defined by [3] Lemme 3.2. By functoriality, these formal schemes again live in a tower over $\mathfrak{X}(\epsilon)$, and they carry compatible $(\mathbb{Z}/p^n\mathbb{Z})^\times$-actions. By [3] Lemme 4.1, $\mathfrak{X}_{r,\mathrm{Ig}(p^n)} = X_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \times_{X_{\mathbb{F}_p}} \mathfrak{X}_r$.

We now consider the limit of the formal Igusa tower in the category of formal schemes:

$$\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)} := \varprojlim_n \mathfrak{X}_{r,\mathrm{Ig}(p^n)}.$$

This is a non-Noetherian flat formal scheme over $\mathbb{F}_p[[t]]$ with a natural $\mathbb{Z}_p^\times$-action.

3.1.3 Frobenius and the canonical subgroup

For an $\mathbb{F}_p$-algebra S, there is over all of X_S a canonical subgroup of the universal semi-abelian scheme, given by the kernel of Frobenius. The relative Frobenius $X_S \to (X_S)^{(p)} = X_S$ can be interpreted in terms of moduli as the map sending an elliptic curve $E|S$ to $E^{(p)} = E/\ker F$.

Over $K = \mathbb{F}_p((t))$, this relative Frobenius is of the form $\mathcal{X}_{r+1} \to \mathcal{X}_r = \mathcal{X}_{r+1}^{(p)}$. This map has a canonical formal model, for example by Lemma 3.1.3 since Frobenius is finite:

$$\phi : \mathfrak{X}_{r+1} \to \mathfrak{X}_r.$$

Alternatively, this can also be seen by direct inspection, or using Proposition 3.3 of [3] which moreover says that ϕ is flat of degree p. We note that for general r, the map ϕ is not itself a relative Frobenius map, just a formal model thereof, since $\mathfrak{X}_{r+1}^{(p)}$ need not be normal.

The same applies to the Igusa curves, and we get formal models of the relative Frobenius

$$\phi : \mathfrak{X}_{r+1,\mathrm{Ig}(p^n)} \to \mathfrak{X}_{r,\mathrm{Ig}(p^n)}.$$

These commute with the forgetful maps in n as well as with the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action, since this is true generically. We may form the "relative perfection" of the $\mathfrak{X}_{r,\mathrm{Ig}(p^n)}$ by iterating ϕ:

$$\mathfrak{X}_\infty := \varprojlim_{\phi,r} \mathfrak{X}_r, \qquad \mathfrak{X}_{\infty,\mathrm{Ig}(p^n)} := \varprojlim_{\phi,r} \mathfrak{X}_{r,\mathrm{Ig}(p^n)}.$$

In the limit over both n and r we obtain the limit of the "relatively perfect Igusa tower"

$$\mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)} := \varprojlim_n \varprojlim_{\phi,r} \mathfrak{X}_{r,\mathrm{Ig}(p^n)} = \varprojlim_n \mathfrak{X}_{\infty,\mathrm{Ig}(p^n)}.$$

We note that these formal schemes are not perfect, only relatively perfect over $\mathbb{F}_p[[t]]$.

To summarise, the main objects whose construction we have recalled in this section can be organised in the following commutative diagram, a $\mathbb{Z}_p^\times$-equivariant morphism of towers

$$\begin{tikzcd} \mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)} \arrow[r] \arrow[d] & \mathfrak{X}_\infty \arrow[d] \\ \mathfrak{X}_{r,\mathrm{Ig}(p^\infty)} \arrow[r] & \mathfrak{X}_r. \end{tikzcd} \tag{13}$$

Remark 3.1.4. From the point of view of the next section, the reason for working with the constructions from this section is that we think of $\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}$ as being a formal model of

$$\mathcal{X}_{\mathrm{Ig}(p^\infty)}(\epsilon) := (\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)})^{\mathrm{ad}}_\eta \sim \varprojlim_n \mathcal{X}_{\mathbb{F}_p((T)),\mathrm{Ig}(p^n)}(\epsilon)$$

in the sense of [52], §2.4. Unfortunately, we do not know whether the adic space $\mathcal{X}_{\mathrm{Ig}(p^\infty)}(\epsilon)$ in the sense of Scholze–Weinstein is again sheafy, i.e. is an adic space in the sense of Huber. One can show based on section 8 that $\mathcal{X}_{\mathrm{Ig}(p^\infty)}(0)$ is sousperfectoid. However, we suspect that this sousperfectoidness might not "overconverge", and we are therefore unsure whether we even expect $\mathcal{X}_{\mathrm{Ig}(p^\infty)}(\epsilon)$ to be sheafy over the locus of supersingular reduction. In contrast, one can show that the generic fibre of $\mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)}$ is indeed sousperfectoid.

We therefore think of the locally ringed space $(\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}, \mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}[1/t])$ as a surrogate for a "missing" infinite level Igusa tower that we are as yet unable to define.

Notwithstanding this technical remark, we think that the constructions of [3] are strikingly elegant, particularly when compared to more classical definitions of overconvergent p-adic modular forms that rely for example on tricks with Eisenstein series.

3.2 The perfectoid setting

We now move on from $\mathbb{F}_p((t))$ to the case of a perfectoid field K over $\mathbb{F}_p((t))$ with ring of integers R. In doing so, we shall also prepare the notation employed in the next section:

We would like to drop R and K from notation. However, for the tilting equivalence in the next sections, we need to distinguish the modular curves in characteristic p from those in characteristic 0. We shall therefore follow the notational convention from [50]: As a general principle, for any p-adic object Z over $\mathbb{Z}_p$, we will denote its t-adic counterpart over $\mathbb{F}_p((t))$ by Z', i.e. by an additional prime in the notation.

We therefore set $\mathfrak{X}' := \mathfrak{X}_R$ and $\mathcal{X}' := \mathcal{X}_K$, and for $0 \leq \epsilon \in \log|K|$, let $\mathcal{X}'(\epsilon) := \mathcal{X}_K(\epsilon)$. Over these, we have an analytic Igusa tower of adic spaces $\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon) := \mathcal{X}_{K,\mathrm{Ig}(p^n)}(\epsilon)$.

3.2.1 The formal model of the Igusa tower

In order to define the formal model of the Igusa tower in this case, we cannot simply follow the discrete case verbatim: Normality of formal schemes is problematic in the non-Noetherian case, and excellence is definitely not satisfied. Instead, we opt for a definition by base-change, a basic strategy which we shall often use in the following:

Like before, let $0 \leq \epsilon \in \log|K|$ and fix an element $t^\epsilon \in K$ with $|t^\epsilon| = |t|^\epsilon$. Then there is a unique morphism $\iota : \mathbb{F}_p[[t]] \to R$ sending $t \mapsto t^\epsilon$, which gives rise to an embedding $\mathbb{F}_p((t)) \hookrightarrow K$. In other words, we now consider two different copies of $\mathbb{F}_p((t))$ inside K.

Since by definition, $\mathcal{X}'(\epsilon) = \mathcal{X}'(|\mathrm{Ha}| \geq |t^\epsilon|)$, we can now reinterpret $\mathcal{X}'(\epsilon)$ as the base change of $\mathcal{X}'_{\mathbb{F}_p((t))}(1)$ along $\iota : \mathbb{F}_p((t)) \to K$. This is the reason we allowed $r = -1$ in the last section. We also obtain a canonical formal model by base-changing $\mathfrak{X}'_{\mathbb{F}_p[[t]]}(1)$:

Definition 3.2.1. With $\iota : \mathbb{F}_p[[t]] \to R$ defined as above, we set for any $n \in \mathbb{N}$:

$$\begin{aligned}\mathfrak{X}'(\epsilon) :=& \mathfrak{X}_{\mathbb{F}_p[[t]]}(1) \times_{\mathbb{F}_p[[t]],\iota} R \\ \mathfrak{X}'_{\mathrm{Ig}(p^n)}(\epsilon) :=& \mathfrak{X}_{\mathbb{F}_p[[t]],\mathrm{Ig}(p^n)}(1) \times_{\mathbb{F}_p[[t]],\iota} R.\end{aligned}$$

Explicitly, like in (12), the formal scheme $\mathfrak{X}'(\epsilon)$ is locally over $U = \mathrm{Spf}(S) \subseteq \mathfrak{X}_R$ of the form $\mathfrak{X}'(\epsilon)_{|U} = \mathrm{Spf}(S\langle X\rangle/(X\mathrm{Ha} - t^\epsilon))$. It therefore coincides with the "$\mathfrak{X}'(\epsilon)$" of III.2 [50].

We can again define a limit of the Igusa tower in the category of formal schemes:

$$\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) = \varprojlim_n \mathfrak{X}'_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathfrak{X}'(\epsilon).$$

It is clear from this definition that for $n \in \mathbb{N}$, the rigid generic fibre of $\mathfrak{X}'_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathfrak{X}'(\epsilon)$ is $\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}'(\epsilon)$. We will see in the next section that the formal schemes we have just defined are in fact again integrally closed in their generic fibre, so that we can *a posteriori* see them as being the "normalisations" of $\mathfrak{X}_R(\epsilon)$ in $\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}'(\epsilon)$, even though this notion is in general problematic in the absence of excellence assumptions.

In particular, the reason we use $\mathfrak{X}_{\mathbb{F}_p[[t]]}(1) = \mathfrak{X}_{-1}$ instead of $\mathfrak{X}_r$ for $r \geq 0$ is that base-change along the latter would not produce an integrally closed formal scheme. Nevertheless, we still have a natural map $\mathfrak{X}'(p^{-k}\epsilon) \to \mathfrak{X}_{\mathbb{F}_p[[t]]}(p^{-k})$, even though this is not the base-change:

Lemma 3.2.2. *The base change $\mathcal{X}'(p^{-k}\epsilon) \to \mathcal{X}_{\mathbb{F}_p[[t]]}(p^{-k})$ has a canonical formal model*

$$\mathfrak{X}'(p^{-k}\epsilon) \to \mathfrak{X}_{k-1} = \mathfrak{X}_{\mathbb{F}_p[[t]]}(p^{-k}).$$

Proof. In the notation of equation (12), this is locally given by the natural map

$$S\langle X\rangle/(X\mathrm{Ha}^{p^k} - t) \to S\hat{\otimes}_\iota \mathcal{O}_K\langle X\rangle/(X\mathrm{Ha}^{p^k} - t^\epsilon) \to S\hat{\otimes}_\iota \mathcal{O}_K\langle X\rangle/(X\mathrm{Ha} - t^{p^{-k}\epsilon})$$

where the first map sends $t \mapsto t^\epsilon$, $X \mapsto X$ and the second sends $X \mapsto X^{p^k}$. $\square$

3.2.2 Frobenius

In contrast to the situation over $\mathbb{F}_p[[t]]$, the Frobenius of $\mathfrak{X}'(p^{-1}\epsilon) \to \mathrm{Spf}(R)$ is of the form

$$\phi : \mathfrak{X}'(p^{-1}\epsilon) \to \mathfrak{X}'(\epsilon).$$

Locally, this follows from the fact that in the notation from the last section

$$(S\langle X\rangle/(X\mathrm{Ha} - t^{\epsilon/p}))^{(p)} = S\langle X\rangle/(X\mathrm{Ha} - t^\epsilon).$$

In particular, ϕ is a formal model for the relative Frobenius of $\mathcal{X}'(\epsilon)$.

Recall from [50], Definition III.2.18 that there is a perfection functor which sends a flat formal scheme $\mathfrak{Y}$ over $\mathbb{F}_p[[t^{1/p^\infty}]]$ to $\mathfrak{Y}^{\mathrm{perf}} := \varprojlim_{F_{\mathrm{abs}}} \mathfrak{Y} = \varprojlim_{F_{\mathrm{rel}}} \mathfrak{Y}^{(p^{-n})}$. One can therefore construct the inverse limit over ϕ simply by applying perfection functors:

$$\mathfrak{X}'(\epsilon)^{\mathrm{perf}} = \varprojlim_\phi \mathfrak{X}'(p^{-n}\epsilon).$$

The same applies to $\mathfrak{X}'_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}$. In analogy to diagram (13), we can summarise this construction in a commutative diagram of formal schemes

$$\begin{tikzcd} \mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \arrow[r] \arrow[d] & \mathfrak{X}'(\epsilon)^{\mathrm{perf}} \arrow[d] \\ \mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) \arrow[r] & \mathfrak{X}'(\epsilon). \end{tikzcd} \tag{14}$$

Lemma 3.2.3. *The generic fibre $\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$ of $\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathfrak{X}'(\epsilon)^{\mathrm{perf}}$ is a pro-étale $\mathbb{Z}_p^\times$-torsor of perfectoid spaces. It is moreover the unique perfectoid tilde-limit*

$$\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \sim \varprojlim \mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}.$$

Proof. That the spaces are perfectoid is clear from Theorem 5.2, [48], since the Frobenius on the formal models is an isomorphism by definition. The tilde-limit description follows from Proposition 2.4.2 of [52]. Uniqueness follows from Proposition 2.4.5, [52] which gives the perfectoid tilde-limit a universal property. To see the statement about torsors, we first note that since $\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}'(\epsilon)$ is an étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor, the same is true for the pullback

$$\mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} = \mathcal{X}'_{\mathrm{Ig}(p^n)}(\epsilon) \times_{\mathcal{X}'(\epsilon)} \mathcal{X}'(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'(\epsilon)^{\mathrm{perf}},$$

where the first isomorphism uses that perfectoid tilde-limits commute with fibre products by the universal property. This also shows the $\mathbb{Z}_p^\times$-torsor property in the limit $n \to \infty$. $\square$

3.3 t-adic modular forms

3.3.1 Definition of t-adic modular forms over $\mathbb{F}_p((t))$

In this section, we define t-adic modular forms following Andreatta–Iovita–Pilloni. However, we slightly modify the definition to avoid a potential issue in the proof of [3], Théorème 4.1.

Let $r_0 := 2$ if $p \geq 3$ or $r_0 := 3$ if $p = 2$. Let $r_0 \leq r \in \mathbb{N}$. In this section, we work with the modular curves over $\mathbb{F}_p[[t]]$ from section 3.1. Let $\varphi : \mathfrak{X}_r \to X_{\mathbb{F}_p}$ be the canonical morphism.

In [3], §4.4, the authors define a sheaf of t-adic modular forms on $\mathfrak{X}_r = \mathfrak{X}_{\mathbb{F}_p[[t]]}(1/p^{r+1})$ for the canonical weight $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ defined by $q \mapsto 1 + t$. We shall denote this sheaf by $\mathfrak{w}_?^{\overline{\kappa}}$, since we do not know if it coincides with the definition we are going to work with later. The definition is as follows: For any $U \subseteq \mathfrak{X}_r$, set

$$\mathfrak{w}_?^{\overline{\kappa}}(U) := \{f \in q_*\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}(U) \mid \gamma^* f = \kappa^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times\}.$$

Since the $\mathbb{Z}_p^\times$-action fixes the subsheaf $\mathcal{O}_{\mathfrak{X}_r} \subseteq q_*\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}$, this is clearly an $\mathcal{O}_{\mathfrak{X}_r}$-module. In Théorème 4.1, the authors show that $\mathfrak{w}_?^{\overline{\kappa}}$ locally on $V \subseteq \mathfrak{X}_r$ has an invertible section $s_V \in \mathcal{O}^\times_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}$. This would imply that $\mathfrak{w}_?^{\overline{\kappa}}$ is invertible if, as claimed in [2], Lemma 7.4.2,

$$(q_*\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}})^{\mathbb{Z}_p^\times} \stackrel{?}{=} \mathcal{O}_{\mathfrak{X}_r}. \tag{15}$$

In the proof of Théorème 4.1, the authors show this locally on the base of $\varphi : \mathfrak{X}_r \to X_{\mathbb{F}_p}$,

$$(\varphi_* q_*\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}})^{\mathbb{Z}_p^\times} = \varphi_*\mathcal{O}_{\mathfrak{X}_r}. \tag{16}$$

Unfortunately, we we fail to see how (16) implies (15). The technical issue is the following:

Question 3.3.1. Let R be a t-adically complete ring for some $t \in R$, let S be a t-adically complete R-algebra and let G be a profinite group acting continuously on S such that $R := S^G$. Let moreover $f \in R$. Then when is it true that we have

$$S\langle f^{-1}\rangle^G = R\langle f^{-1}\rangle?$$

The obstruction for commuting $(-)^G$, reduction mod t^n and $\varprojlim_n$ is precisely

$$\varprojlim_{n\to\infty} (H^1(G, S)[t^n] \otimes_R R[f^{-1}]).$$

In our specific example, we can think of the topological space underlying $\mathfrak{X}(\epsilon)$ as being $X_{\mathbb{F}_p}$ with copies of the line $\mathbb{A}^1_{\mathbb{F}_p}$ glued into each of the finitely many points where $\mathrm{Ha} = 0$. From this topological perspective, the question is what happens on each of the lines.

To circumvent this question, we opt to slightly modify the definition of $\mathfrak{w}_?^{\overline{\kappa}}$ in a way that ensures we get a line bundle, regardless of whether equation (15) holds true:

Definition 3.3.2. We define a sheaf $\mathfrak{w}^{\overline{\kappa}}$ on $\mathfrak{X}_r$ as follows: Recall that $X_{\mathbb{F}_p}$ denotes the compactified modular curve over $\mathbb{F}_p$, and $\varphi : \mathfrak{X}_r \to X_{\mathbb{F}_p}$ denotes the canonical morphism. Consider the ringed space $\varphi_*\mathfrak{X}_r := (X_{\mathbb{F}_p}, \varphi_*\mathcal{O}_{\mathfrak{X}_r})$. Then there is a canonical morphism of ringed spaces $\psi : \mathfrak{X}_r \to \varphi_*\mathfrak{X}_r$. We first define a module on $\varphi_*\mathfrak{X}_r$ by setting

$$\mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}} := \{f \in \varphi_* q_*\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}} \mid \gamma^* f = \kappa^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times\}.$$

We then define the sheaf of t-adic modular forms on $\mathfrak{X}_r$ to be $\mathfrak{w}^{\overline{\kappa}} = \psi^*\mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}}$.

Lemma 3.3.3. *Let $r_0 = 2$ if $p > 2$ and $r_0 = 3$ if $p = 2$. Let $r \geq r_0$.*

1. *The sheaf $\mathfrak{w}^{\overline{\kappa}}$ is a line bundle on $\mathfrak{X}_r = \mathfrak{X}_{\mathbb{F}_p[[t]]}(1/p^{r+1})$. It is trivial on the pullback of any open $U \subseteq X_{\mathbb{F}_p}$ on which ω is trivial.*

2. *For any open $V \subseteq \mathfrak{X}_r$ of the form $V = \varphi^{-1}(U)$ for some open $U \subseteq X_{\mathbb{F}_p}$, we have*

$$\mathfrak{w}^{\overline{\kappa}}(V) = \{f \in q_*\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}(V) \mid \gamma^* f = \overline{\kappa}^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times\}.$$

In particular, we have $\varphi_*\mathfrak{w}^{\overline{\kappa}} = \mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}} = \varphi_*\mathfrak{w}^{\overline{\kappa}}_?$, but we suspect that $\mathfrak{w}^{\overline{\kappa}} \subsetneq \mathfrak{w}^{\overline{\kappa}}_?$ on $\mathfrak{X}_r$.

Proof. For the first part, it suffices to see that $\mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}}$ is locally free of rank 1 as an $\varphi_*\mathcal{O}_{\mathfrak{X}_r}$-module on $\varphi_*\mathfrak{X}_r$. For this we note that for any U as in the Lemma, the construction in [3], Théorème 4.1 gives a section $s_V \in \mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}}(U)$ that is invertible as a function on $\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}$. Equation (16) then shows that $\mathfrak{w}^{\kappa}_{\mathrm{pre}}|_U = s \cdot \varphi_*\mathcal{O}_{\mathfrak{X}_r}|_U$ is free of rank 1, as desired.

Part 2 amounts to saying that $\varphi_*\mathfrak{w}^{\overline{\kappa}} = \mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}}$. It suffices to see this locally on $X_{\mathbb{F}_p}$. If U is small enough such that ω is trivial, part 1 shows that we have $\mathfrak{w}^{\overline{\kappa}}|_V = s\mathcal{O}_V$, so that

$$\varphi_*\mathfrak{w}^{\overline{\kappa}}|_U = s \cdot \varphi_*\mathcal{O}_{\mathfrak{X}_r}|_U = \mathfrak{w}^{\overline{\kappa}}_{\mathrm{pre}}|_U. \qquad \square$$

Definition 3.3.4. Let $\omega^{\overline{\kappa},+} := h^*\mathfrak{w}^{\overline{\kappa}}$ for the morphism of ringed spaces $h : (\mathcal{X}_r, \mathcal{O}^+_{\mathcal{X}_r}) \to \mathfrak{X}_r$ and let $\omega^{\overline{\kappa}} = \omega^{\overline{\kappa}}[1/t]$. These are the analytic sheaves of (integral) t-adic modular forms.

Definition 3.3.5. For any $r_0 \leq r \in \mathbb{Z} \cup \{\infty\}$ and $\epsilon = p^{-r}$, let $M^+_{\overline{\kappa}}(\epsilon) := \Gamma(\mathfrak{X}_{\mathbb{F}_p[[t]]}(\epsilon), \mathfrak{w}^{\overline{\kappa}})$.

3.3.2 Definition of t-adic modular forms over more general rings

In this section we modify the definition of the last section in two ways which are closely related: First, we replace $\mathbb{F}_p((t))$ by a perfectoid field extension K. In this section we could more generally work over any non-archimedean field extension of $\mathbb{F}_p((t))$, but we shall restrict to the case of perfectoid fields for simplicity. The second goal is to replace the canonical character $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ by *any* nontrivial continuous character $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$. As we shall discuss, both goals are essentially just a matter of base-change.

Definition 3.3.6. Let K be a perfectoid field extension of $\mathbb{F}_p((t))$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any nontrivial weight. Recall that $T_\kappa := \kappa(q) - 1 \in \mathcal{O}_K$. We regard T_κ as the pseudo-uniformiser defined by κ. We moreover introduce the following notation that we will use throughout:

1. Let $\iota = \iota_\kappa : \mathbb{F}_p[[t]] \to \mathcal{O}_K$ be the map sending $t \mapsto T_\kappa$.
2. Let $\varepsilon \geq 0$ be such that $|T_\kappa| = |t|^\varepsilon$. We set $\epsilon_\kappa := p^{-r_0-1}\varepsilon$, where r_0 is as in Lemma 3.3.3.
3. With this setup, we define $\mathfrak{w}^\kappa$ to be the line bundle on $\mathfrak{X}'(\epsilon)$ obtained by base change of $\mathfrak{w}^{\overline{\kappa}}$ along the morphism $\mathfrak{X}'(\epsilon) = \mathfrak{X}'(p^{-r-1}\varepsilon) \to \mathfrak{X}_r$ associated to ι by Lemma 3.2.2. We sometimes also allow $\kappa = 1$, in which case for any $\epsilon > 0$ we set $\mathfrak{w}^\kappa = \mathcal{O}_{\mathfrak{X}'(\epsilon)}$.
4. We define $M^+_\kappa(\epsilon) := \Gamma(\mathfrak{X}'(\epsilon), \mathfrak{w}^\kappa)$ and $M_\kappa(\epsilon) := M^+_\kappa(\epsilon)[1/t]$.

The following Lemma says that we could equivalently define $\mathfrak{w}^\kappa$ more intrinsically, namely without explicit reference to $\mathbb{F}_p[[t]]$ and $\overline{\kappa}$, as a sheaf of invariants, like in the last section.

Lemma 3.3.7. *For any open $U \subseteq X_{\mathbb{F}_p}$ with pullback $V \subseteq \mathfrak{X}'(\epsilon)$, we have*

$$\mathfrak{w}^\kappa(V) = \{f \in q_*\mathcal{O}_{\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)}(V) \mid \gamma^* f = \kappa^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times\}.$$

Proof. Like for Lemma 3.3.3.2, it suffices to prove this locally on a trivialising cover, so that we may assume that U is affine and ω is trivial on U. It then suffices to prove that

$$\mathcal{O}_{\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)}(V)^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}'(\epsilon)}(V).$$

Write $R := \mathbb{F}_p[[t]]$, $S := \mathcal{O}_K$, and $A := \mathcal{O}_{\mathfrak{X}_{\mathbb{F}_p[[t]]}(1)}(U)$, $B := \mathcal{O}_{\mathfrak{X}_{\mathbb{F}_p[[t]],\mathrm{Ig}(p^\infty)}(1)}(U)$. We have to show $(B\hat{\otimes}_R S)^{\mathbb{Z}_p^\times} = A\hat{\otimes}_R S$. Since R is discretely valued, this holds by Lemma A.3.11. $\square$

3.4 t-adic perfectoid modular forms

In this subsection, we define perfectoid t-adic modular forms. The definition is essentially analogous to that of t-adic modular forms, by replacing the respective Igusa towers with their relatively perfect versions. However, the perfect setting actually simplifies the construction in two ways: First, we may work with analytic adic spaces, since the issue raised in Remark 3.1.4 disappears: The (relatively) perfected Igusa tower is clearly (sous-)perfectoid. Second, the issue regarding invariants raised in Question 3.3.1 disappears, as we shall now discuss. These are arguably two examples for how perfectoid modular forms are often technically slightly easier to deal with then their unperfected counterparts.

Lemma 3.4.1. *In the following, all sheaves are tacitly pushed forward to the indicated base:*

1. *We have* $(\mathcal{O}_{\mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)}})^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}_\infty}$ *as sheaves on* $\mathfrak{X}_\infty$.
2. *We have* $(\mathcal{O}_{\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}})^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}'(\epsilon)^{\mathrm{perf}}}$ *as sheaves on* $\mathfrak{X}'(\epsilon)^{\mathrm{perf}}$.
3. *We have* $(\mathcal{O}^+_{\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}})^{\mathbb{Z}_p^\times} = \mathcal{O}^+_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}$ *as sheaves on* $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$.

In particular, the analogous statement in 3 also holds for $\mathcal{O}^+$ *replaced by* $\mathcal{O}$.

Proof. The first part is the case of $I = \{\infty\}$ of Proposition 6.3 in [3]. In contrast to the discussion around equation (16), the proof does not rely on a computation locally on $X_{\mathbb{F}_p}$ but locally on $\mathfrak{X}_\infty$, and uses that $H^1(G, \mathcal{O}_{\mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)}}(U))$ is annihilated by t for any affine $U \subseteq \mathfrak{X}_\infty$. This, however, in turn shows that the issue raised in Question 3.3.1 would not arise here anyway since formal localisation then commutes with invariants by Lemma A.3.11.2.

For the second part, we cannot simply argue by base-change, since $\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ is *not* just the base change of $\mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)}$. Instead, we could simply copy the strategy for 1, which carries over since one can show using the exact same methods that $H^1(G, \mathcal{O}_{\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)})$ is t-torsion – in fact, even almost zero. We shall not spell this out in detail, since alternatively, we will see in Proposition 4.1.2.2 that part 2 can be deduced from the third part.

Part 3 follows like in Lemma 2.1.3.2: The morphism $\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$ is a pro-étale $\mathbb{Z}_p^\times$-torsor of perfectoid spaces by Lemma 3.2.3, and the integral structure presheaf $\mathcal{O}^+$ on the pro-étale site is a sheaf by [51], Proposition 8.5.(iii). $\square$

Definition 3.4.2.

1. Over $\mathbb{F}_p[[t]]$, let $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ be the canonical boundary character, then we define the sheaf of perfectoid t-adic modular forms on $\mathfrak{X}_\infty$ to be
$$\mathfrak{w}^{\overline{\kappa},\mathrm{perf}} := \mathcal{O}_{\mathfrak{X}_{\infty,\mathrm{Ig}(p^\infty)}}[\overline{\kappa}^{-1}].$$
2. Over $\mathcal{O}_K$, let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight, then we define a sheaf on $\mathfrak{X}'(\epsilon)^{\mathrm{perf}}$ by
$$\mathfrak{w}^{\kappa,\mathrm{perf}} := \mathcal{O}_{\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}}[\kappa^{-1}].$$
3. Over $\mathcal{O}_K$, we define analytic sheaves of t-adic perfectoid modular forms on $\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$
$$\omega^{\kappa,+,\mathrm{perf}} := \mathcal{O}^+_{\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}}[\kappa^{-1}], \quad \omega^{\kappa,\mathrm{perf}} = \omega^{\kappa,+,\mathrm{perf}}[1/t].$$
4. The (integral) spaces of t-adic perfectoid modular forms of weight κ are $M_\kappa^{+,\mathrm{perf}}(\epsilon) := \Gamma(\mathcal{X}'(\epsilon)^{\mathrm{perf}}, \omega^{\kappa,+,\mathrm{perf}})$ and $M_\kappa^{\mathrm{perf}}(\epsilon) := \Gamma(\mathcal{X}'(\epsilon)^{\mathrm{perf}}, \omega^{\kappa,\mathrm{perf}})$. It is clear from the definition that we have $M_\kappa^+(\epsilon) \subseteq M_\kappa^{+,\mathrm{perf}}(\epsilon)$, so every t-adic modular form is a t-adic perfectoid modular form. If we wish to emphasize that a t-adic perfectoid modular form is already contained in $M_\kappa^+(\epsilon)$, we shall call it a "true t-adic modular form".

   Finally, for any $r_0 \leq r \in \mathbb{Z}\cup\{\infty\}$, let $M_{\overline{\kappa}}^{+,\mathrm{perf}}(1/p^{r+1}) := \Gamma(\mathfrak{X}_{\mathbb{F}_p[[t]]}(1/p^{r+1})^{\mathrm{perf}}, \mathfrak{w}^{\overline{\kappa},\mathrm{perf}})$.

Proposition 3.4.3. *Let $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ and $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be weights like above.*

1. *The sheaf $\mathfrak{w}^{\overline{\kappa},\mathrm{perf}}$ is the pullback of $\mathfrak{w}^{\overline{\kappa}}$ along $\mathfrak{X}_\infty \to \mathfrak{X}_r$.*

2. *The sheaf $\mathfrak{w}^{\kappa,\mathrm{perf}}$ is the pullback of $\mathfrak{w}^{\kappa}$ along $\mathfrak{X}'(\epsilon)^{\mathrm{perf}} \to \mathfrak{X}'(\epsilon)$.*

3. *The sheaf $\omega^{\kappa,+,\mathrm{perf}}$ is the pullback of $\omega^{\kappa,+}$ along $\mathcal{X}'(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'(\epsilon)$.*

In particular, these sheaves are all invertible modules on their respective ringed spaces.

Proof. These are all application of the base change Lemma 2.8.4: The first part is [3] Proposition 6.6. It follows from applying the Lemma to diagram (13).

Part 2 follows from applying the Lemma to diagram (14). The third part follows from the generic fibre of the same diagram, where for technical reasons we need this to mean the map of locally ringed spaces $(\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon), \mathcal{O}[1/t]) \to (\mathfrak{X}'(\epsilon), \mathcal{O}[1/t])$ for the bottom row. $\square$

3.5 q-expansions

There is also a good theory of q-expansions of functions on modular curves over K, which is analogous to the theory for p-adic modular curves. Again we give a brief exposition of the main results and refer to [27], §4 for details and proofs. Like in the p-adic case, we assume for simplicity that there is a primitive N-th unit root $\zeta_N \in K$ where N is the tame level.

Denote by $D' := \mathrm{Spf}(\mathcal{O}_K[[q]])^{\mathrm{ad}}_\eta$ the open unit disc over K, where $\mathcal{O}_K[[q]]$ is endowed with the (t,q)-adic topology. Then around any cusp c of the tame level modular curve $\mathcal{X}'$, there is a canonical open immersion $D' \hookrightarrow \mathcal{X}'$ which sends the origin $0 \in D'$ to that cusp. Again D' arises as the parameter space for Tate curves equipped with level structure.

Passing to perfections, we get for any cusp an open immersion

$$D'_\infty := D'^{\mathrm{perf}} \hookrightarrow \mathcal{X}'(\epsilon)^{\mathrm{perf}}$$

where $D'_\infty = \mathrm{Spf}(\mathcal{O}_K[[q^{1/p^\infty}]])^{\mathrm{ad}}_\eta$ is the perfectoid open unit disc. Like in the p-adic case, $\mathcal{O}(D')$ and $\mathcal{O}(D'_\infty)$ are rings of formal power series with convergence conditions contained in $K[[q]]$ and $K[[q^{1/p^\infty}]]$, and the rings of bounded functions are simply

$$\mathcal{O}^+(D) = \mathcal{O}_K[[q]] \text{ and } \mathcal{O}^+(D_\infty) = \mathcal{O}_K[[q^{1/p^\infty}]].$$

In the Igusa tower, we instead obtain over any cusp c of $\mathcal{X}'$ for any $n \in \mathbb{N}$ a $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant open immersion $\underline{\mathbb{Z}/p^n\mathbb{Z}} \times D' \hookrightarrow \mathcal{X}'_{\mathrm{Ig}(p^n)}$. In the limit $n \to \infty$, this gives rise over any cusp of $\mathcal{X}'$ to an open immersion of pro-étale $\mathbb{Z}_p^\times$-torsors

$$\underline{\mathbb{Z}_p^\times} \times D' \hookrightarrow \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0)$$

and on perfections we similarly get $\underline{\mathbb{Z}_p^\times} \times D'_\infty \hookrightarrow \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}$. These have functions

$$\mathcal{O}^+(\underline{\mathbb{Z}_p^\times} \times D'_\infty) = \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]])$$

and similarly for $\underline{\mathbb{Z}_p^\times} \times D'$. We can thus define q-expansions as in the p-adic case:

Definition 3.5.1. Let f be a function on $\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$. Then for any cusp $c \in \mathcal{X}'$, we define the q-expansion of f at c to be the image f_c of f under the morphism

$$\mathcal{O}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}) \to \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]])[1/t]$$

given by the map $\mathbb{Z}_p^\times \times D'_\infty \hookrightarrow \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ associated to c. Like in Lemma 2.3.2 we see:

Lemma 3.5.2. *Let c_0 be a cusp of $\mathcal{X}'(\epsilon)$ and let c be any cusp of $\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ over c_0. Let $\gamma \in \mathbb{Z}_p^\times$, giving another cusp γc. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight. Then for any $f \in M_\kappa^{\mathrm{perf}}(\epsilon)$,*

$$f_{\gamma c} = \kappa^{-1}(\gamma) f_c.$$

In particular, up to a scalar, the q-expansion of f over c only depends on c_0.

Proposition 3.5.3 (q-expansion principle I, [27], Proposition 6.1)**.** *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight and let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then for any $f \in M_\kappa^{\mathrm{perf}}(\epsilon)$, the following are equivalent:*

1. *$f = 0$.*

2. *The q-expansion $f_c \in \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_{K'}[[q^{1/p^\infty}]])[1/t]$ vanishes at all cusps c*

Proposition 3.5.4 (q-expansion principle II, [27], Corollary 6.10)**.** *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight and let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then for any $f \in M_\kappa^{\mathrm{perf}}(\epsilon)$, the following are equivalent:*

1. *f is already contained in the subspace $M_\kappa(\epsilon) \subseteq M_\kappa^{\mathrm{perf}}(\epsilon)$.*

2. *The q-expansions f_c are already in $\mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_{K'}[[q]])[1/t]$ for all cusps c.*

Remark 3.5.5. In the next section, we are also going to define t-adic modular forms of some "level" $\Gamma_0(p^n)$. In analogy with Proposition 2.3.4, one can generalise Proposition 3.5.4 to say that these intermediate forms are precisely those with q-expansions in $\mathcal{O}_{K'}[[q^{1/p^n}]]$.

3.6 Frobenius operators and trace maps

In this section, we discuss various instances of Frobenius operators on modular forms. Conceptually, these replace the p-adic Atkin–Lehner isomorphism in the t-adic setting. In particular, we will need the relative Frobenius for the definition of the U_p-operator.

For the definition, we will use the morphism of correspondences

$$\begin{tikzcd}
\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) \arrow[d] & \mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(p^{-1}\epsilon) \arrow[l, "i_\infty"'] \arrow[r, "\phi_\infty"] \arrow[d] & \mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) \arrow[d] \\
\mathfrak{X}'(\epsilon) & \mathfrak{X}'(p^{-1}\epsilon) \arrow[l, "i"'] \arrow[r, "\phi"] & \mathfrak{X}'(\epsilon)
\end{tikzcd}$$

where i denotes the formal model of the restriction, the map ϕ is the relative Frobenius from §3.2.2, and the maps i_∞ and ϕ_∞ are the respective versions of these maps that we get in the limit in the Igusa tower. Since these maps are $\mathbb{Z}_p^\times$-equivariant, we see from Lemma 2.8.4:

Lemma 3.6.1. *There is a canonical "relative Frobenius" isomorphism $F : \phi^*\mathfrak{w}^\kappa = i^*\mathfrak{w}^\kappa$.*

Definition 3.6.2. We denote by $M^+_{\kappa,\Gamma_0(p^n)}(\epsilon) := \Gamma(\mathfrak{X}'(p^{-n}\epsilon), \phi^{n*}\mathfrak{w}^\kappa)$ the global sections of $\phi^{n*}\mathfrak{w}^\kappa$. Let $M_{\kappa,\Gamma_0(p^n)}(\epsilon) := \Gamma(\mathcal{X}'(p^{-n}\epsilon), \phi^{n*}\omega^\kappa)$.

Let us renormalise the q-expansions of $\mathcal{X}(p^{-n}\epsilon)$ via $q \mapsto q^{1/p^n}$ to be in $\mathcal{O}_K[[q^{1/p^n}]]$. This is so that ϕ is on q-expansions the inclusion $\mathcal{O}_K[[q]] \hookrightarrow \mathcal{O}_K[[q^{1/p}]]$. We may then think of elements of $M^+_{\kappa,\Gamma_0(p^n)}(p^{-n}\epsilon)$ as having q-expansions in $\mathcal{O}_K[[q^{1/p^n}]]$.

Comparing with Proposition 3.4.3.2, we may think of $M_\kappa^{\mathrm{perf}}(\epsilon)$ as being "$M_{\kappa,\Gamma_0(p^\infty)}(\epsilon)$".

The relative Frobenius from Lemma 3.6.1 on global sections defines a map

$$F^n : M^+_{\kappa,\Gamma_0(p^n)}(\epsilon) \xrightarrow{\sim} M^+_\kappa(p^{-n}\epsilon) \tag{17}$$

We think of F^n as a t-adic analogue of the p-adic Atkin–Lehner isomorphism. This is also the reason for the notation $M^+_{\kappa,\Gamma_0(p^n)}(\epsilon)$ in analogy with characteristic 0.

Remark 3.6.3. One can check that on q-expansions, F^n sends $\sum a_m q^{m/p^n} \mapsto \sum a_m q^m$.

There is also an absolute Frobenius morphism, as well as a base change along Frobenius:

Lemma 3.6.4. *1. The absolute Frobenius $F^n_{\mathrm{abs}} : \mathfrak{X}'(\epsilon) \to \mathfrak{X}'(\epsilon)$ induces an isomorphism $F^{n*}_{\mathrm{abs}}\mathfrak{w}^\kappa = \mathfrak{w}^{\kappa^{p^n}}$. Its adjoint defines a map $M^+_\kappa(\epsilon) \to M^+_{\kappa^{p^n}}(\epsilon)$, $f \mapsto f^{p^n}$.*

2. The base change $b^n : \mathfrak{X}'(p^n\epsilon) \xrightarrow{\sim} \mathfrak{X}'(\epsilon)$ along the absolute Frobenius F^n_{abs} of $\mathcal{O}_K$ induces an isomorphism $\mathfrak{w}^\kappa \to b^n_*\mathfrak{w}^{\kappa^{p^n}}$ of sheaves on $\mathfrak{X}'(\epsilon)$. On global sections, this defines an $F^n_{\mathrm{abs}} : \mathcal{O}_K \to \mathcal{O}_K$-linear isomorphism $b^n : M^+_\kappa(\epsilon) \xrightarrow{\sim} M^+_{\kappa^{p^n}}(p^n\epsilon)$.

Proof. The second part follows from the base change diagram of $\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) \to \mathfrak{X}'(\epsilon)$ along $F^n : \mathcal{O}_K \to \mathcal{O}_K$ by Lemma 2.8.4, using that b^n is an isomorphism since K is perfect. The first part then follows from replacing ϵ by $p^{-n}\epsilon$ in part 2, so that $F^{n*}_{\mathrm{abs}}\mathfrak{w}^\kappa = b^{n*}\phi^{n*}\mathfrak{w}^\kappa = \mathfrak{w}^{\kappa^{p^n}}$. $\square$

Remark 3.6.5. One can check that on q-expansions, b^n sends $\sum a_m q^m \mapsto \sum a_m^{p^n} q^m$.

A very important feature of the relative Frobenius map is that it admits a section. This in particular allows us to go back from perfectoid modular forms to true t-adic modular forms, as we shall now discuss. The following will be proved in Corollary 8.1.4 below:

Proposition 3.6.6. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $\delta := 3\epsilon p^3/(p-1)$. Then $\phi^n : \mathfrak{X}'(p^{-n}\epsilon) \to \mathfrak{X}'(\epsilon)$ admits $\mathcal{O}_K$-linear sections $\mathrm{tr}_{\phi^n} : \phi^n_*\mathcal{O}_{\mathfrak{X}'(p^{-n}\epsilon)} \to t^{-\delta}\mathcal{O}_{\mathfrak{X}'(\epsilon)}$ of the map $\mathcal{O}_{\mathfrak{X}'(\epsilon)} \to \mathcal{O}_{\mathfrak{X}'(p^{-n}\epsilon)}$, compatible for varying n. In the limit, we also obtain a section for $\phi^\infty : \mathfrak{X}'(\epsilon)^{\mathrm{perf}} \to \mathfrak{X}'(\epsilon)$, which is also of the form $\mathrm{tr}_{\phi^\infty} : \phi^\infty_*\mathcal{O}_{\mathfrak{X}'(\epsilon)^{\mathrm{perf}}} \to t^{-\delta}\mathcal{O}_{\mathfrak{X}'(\epsilon)}$.*

Remark 3.6.7. One can check that on q-expansions, tr_{ϕ^n} sends $\sum a_m q^{m/p^n} \mapsto \sum a_{mp^n} q^m$.

Corollary 3.6.8. *Upon tensoring with $\mathfrak{w}^\kappa$, tr_{ϕ^n} induces an $\mathcal{O}_K$-linear section*

$$\mathrm{tr}_{\phi^n} : M^+_{\kappa,\Gamma_0(p^n)}(\epsilon) \to t^{-\delta}M^+_\kappa(\epsilon).$$

Similarly, for $\phi^\infty : \mathfrak{X}'(\epsilon)^{\mathrm{perf}} \to \mathfrak{X}'(\epsilon)$, we obtain an $\mathcal{O}_K$-linear section

$$\mathrm{tr} : M^{+,\mathrm{perf}}_\kappa(\epsilon) \to t^{-\delta}M^+_\kappa(\epsilon).$$

Proof. By Proposition 3.4.3, we have $\phi^{\infty*}\mathfrak{w}^\kappa = \mathfrak{w}^{\kappa,\mathrm{perf}}$. The trace map $\phi^\infty_*\phi^{\infty*}\mathcal{O}_{\mathfrak{X}'(\epsilon)} \to t^{-\epsilon}\mathcal{O}_{\mathfrak{X}'(\epsilon)}$ therefore induces a map $\phi^\infty_*\mathfrak{w}^{\kappa,\mathrm{perf}} \to t^{-\epsilon}\mathfrak{w}^\kappa$. The second part of the Corollary follows on global sections. The first part can be seen in the same way using ϕ^n instead. $\square$

3.7 Hecke operators

3.7.1 The tame Hecke algebra

For any prime l that does not divide pN, one can define a Hecke operator T_l exactly like in the p-adic case: Like in the p-adic case, we start with a Cartesian diagram of correspondences

$$\begin{array}{ccccc}
\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) & \xleftarrow{\pi_{2,\infty}} & \mathfrak{X}'_{\mathrm{Ig}(p^\infty)\times\Gamma_0(l)}(\epsilon) & \xrightarrow{\pi_{1,\infty}} & \mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{X}'(\epsilon) & \xleftarrow{\pi_2} & \mathfrak{X}'_{\Gamma_0(l)}(\epsilon) & \xrightarrow{\pi_1} & \mathfrak{X}'(\epsilon).
\end{array} \tag{18}$$

Here the additional $\Gamma_0(l)$ in the index indicates that the respective spaces have a factor of $\Gamma_0(l)$ added to their tame level structure, which relatively represents the choice of a subgroup $D \subseteq E[p^l]$ of the elliptic curve E. The map π_1 then sends $(E, D) \mapsto E$, whereas π_2 sends $(E, D) \mapsto E/D$. Both of these extend canonically to the Igusa curves.

Since $\pi_{1,\infty}$ and $\pi_{2,\infty}$ are clearly $\mathbb{Z}_p^\times$-equivariant, it is immediate from Lemma 2.8.4 that there is a canonical isomorphism $\pi_{1,\infty}^*\mathfrak{w}^\kappa = \pi_{2,\infty}^*\mathfrak{w}^\kappa$. We can thus define T_l by

$$T_l : \Gamma(\mathfrak{w}^\kappa) \to \Gamma(\pi_1^*\mathfrak{w}^\kappa) \xrightarrow{\sim} \Gamma(\pi_2^*\mathfrak{w}^\kappa) \xrightarrow{\kappa(l)\bar{l}^{-1}\,\mathrm{Tr}} \Gamma(\mathfrak{w}^\kappa).$$

where in the last map Tr is the trace of the finite locally free morphism π_1, and where we use that l is coprime to p to make sense of $\kappa(l)$ and of the reduction $\bar{l} \in \mathbb{F}_p^\times$ of l.

3.7.2 The U_p-operator

The U_p-operator can be defined like in [3], 4.4.7, using the relative Frobenius F from (17) and the trace tr_ϕ from Corollary 3.6.8. Let $i : M_\kappa(\epsilon) \hookrightarrow M_\kappa(p^{-1}\epsilon)$ be the restriction.

Definition 3.7.1. We define the U_p-operator to be the composition

$$U_p : M_\kappa(\epsilon) \xrightarrow{i} M_\kappa(p^{-1}\epsilon) \xrightarrow{F^{-1}} M_{\kappa,\Gamma_0(p)}(\epsilon) \xrightarrow{\mathrm{tr}_\phi} M_\kappa(\epsilon).$$

The remarks on q-expansions combine to show that U_p sends $\sum a_n q^n \mapsto \sum a_{np}q^n$ as usual.

Remark 3.7.2. Two remarks are in order: First, like in the p-adic case, while the definition of the Hecke operators in the tame case has a straightforward analogue for perfectoid modular forms, obtained by perfecting diagram (18), the definition of the U_p-operator does not carry over, because ϕ becomes an isomorphism upon perfecting, which makes the analogue of the above definition vacuous. Instead of using Hecke correspondences, one could instead force a definition by composing the above U_p-operator with the trace map $\mathrm{tr} : M_\kappa^{\mathrm{perf}}(\epsilon) \to M_\kappa(\epsilon)$.

Second, we note that while the tame Hecke algebra is defined at the integral level, the non-integral nature of the trace map keeps us from obtaining an integral U_p-operator on the space of overconvergent modular forms. However, the failure to be integral can be controlled and is "small". Also, this failure does not occur for $\epsilon = 0$, i.e. for the convergent forms, which ensures that all Hecke eigenvalues are integral.

Definition 3.7.3. For later reference in the tilting isomorphism, we define the integral Hecke algebra for κ to be the free commutative $\mathbb{Z}_p$-algebra generated by the T_l for primes l coprime to pN and by U_p°. The integral Hecke algebra has a natural action on $M_\kappa^+(\epsilon)$, where we let the operator U_p° act as $T_\kappa^\delta \cdot U_p$ for $\delta = 3/p^{r_0}(p-1)$. This normalisation is chosen so that all operators respect the integral subspaces, in a way that is compatible in families.

4 Integral modular forms and base change

In the last two sections, we have in each characteristic defined a large number of different sheaves of modular forms: In both the p-adic case and the t-adic case we have sheaves ω and ω^+ and perfectoid versions thereof. In characteristic p, we also have formal models $\mathfrak{w}$ and $\mathfrak{w}^{\mathrm{perf}}$ of these sheaves, and these come in two versions, namely either over discretely valued or perfectoid fields. We have already compared these definitions to other definition in the literature. To complement this, the goal of this section is to compare these bundles to each other: In the t-adic case, we show that the integral sheaves $\mathfrak{w}$ and ω^+ are essentially the same, except that the second lives on a larger topological space. This observation also allows us to give a definition of the formal sheaf $\mathfrak{w}$ in the p-adic case.

Secondly, we compare modular forms over different fields via base-change.

4.1 The formal modular curves are normal

In the setting of Andreatta–Iovita–Pilloni, we have the following straightforward comparison between the two different integral sheaves of modular forms

Lemma 4.1.1. *Let $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ be the canonical weight. Let $s : (\mathcal{X}_r, \mathcal{O}^+_{\mathcal{X}_r}) \to (\mathfrak{X}_r, \mathcal{O}_{\mathfrak{X}_r})$ be the natural map of ringed spaces. Then the natural inclusion $\mathfrak{w}^{\overline{\kappa}} \subseteq s_*\omega^{\overline{\kappa},+}$ is an equality.*

Proof. Since $\mathfrak{X}_r$ is normal by Lemma 3.1.3, we have $\mathcal{O}_{\mathfrak{X}_r} = s_*\mathcal{O}^+_{\mathcal{X}_r}$ on $\mathfrak{X}_r$ by Lemma A.2.5. Consequently, the adjoint map $\mathfrak{w}^{\overline{\kappa}} \to s_*\omega^{\overline{\kappa},+}$ to the equality $s^*\mathfrak{w}^{\overline{\kappa}} = \omega^{\overline{\kappa},+}$ is an isomorphism locally on any open $U \subseteq \mathfrak{X}_r$ where $\mathfrak{w}^{\overline{\kappa}}$ is trivial, and thus is an isomorphism globally. □

In this section, we wish to prove an analogous statement for p-adic and t-adic modular forms in the perfectoid setting. For this, we need to replace Lemma 3.1.3 by the following:

Proposition 4.1.2. *Let K be a perfectoid field of characteristic p. Let $\epsilon \geq 0$ with $\epsilon \in \log|K|$.*

1. *The sheaf of rings $\mathcal{O}_{\mathfrak{X}'(\epsilon)}$ is integrally closed in the sheaf $\mathcal{O}_{\mathfrak{X}'(\epsilon)}[1/t]$ on $\mathfrak{X}'(\epsilon)$. In particular, for any open $U \subseteq \mathfrak{X}'(\epsilon)$, we have*
$$\mathcal{O}^+_{\mathcal{X}'(\epsilon)}(U) = \mathcal{O}_{\mathfrak{X}'(\epsilon)}(U).$$

2. *The sheaf of rings $\mathcal{O}_{\mathfrak{X}'(\epsilon)^{\mathrm{perf}}}$ is integrally closed in the sheaf $\mathcal{O}_{\mathfrak{X}'(\epsilon)^{\mathrm{perf}}}[1/t]$ on $\mathfrak{X}'(\epsilon)^{\mathrm{perf}}$. For any open $U \subseteq \mathfrak{X}'(\epsilon)^{\mathrm{perf}}$, we have an honest (not just almost) equality*
$$\mathcal{O}^+_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}(U) = \mathcal{O}_{\mathfrak{X}'(\epsilon)^{\mathrm{perf}}}(U).$$

Proposition 4.1.3. *Let K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $\frac{1}{2} > \epsilon \geq 0$ with $\epsilon \in \log|K|$.*

1. *The sheaf of rings $\mathcal{O}_{\mathfrak{X}(\epsilon)}$ is integrally closed in the sheaf $\mathcal{O}_{\mathfrak{X}(\epsilon)}[1/p]$ on $\mathfrak{X}(\epsilon)$. In particular, for any open $U \subseteq \mathfrak{X}(\epsilon)$, we have*
$$\mathcal{O}^+_{\mathcal{X}(\epsilon)}(U) = \mathcal{O}_{\mathfrak{X}(\epsilon)}(U).$$

2. *The sheaf of rings $\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}$ is integrally closed in the sheaf $\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}[1/p]$ on $\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$. In particular, for any open $U \subseteq \mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$, we have an honest equality*
$$\mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}(U) = \mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}(U).$$

For the proof we need to argue differently than [3] does for Lemma 3.1.3 since the perfectoid base puts us into a non-Noetherian setting, in which we cannot argue by excellence or Serre's criterion. Instead, our strategy is as follows: We first consider the perfectoid setting, where the analogous statement is true in the almost category. We then use:

Proposition 4.1.4. *Let K be a perfectoid field with pseudo-uniformiser $\pi \in \mathcal{O}_K$. Let A be a flat $\mathcal{O}_K$-algebra of topologically finite presentation. Assume that there is a map $h : A \to A_\infty$ into a flat $\mathcal{O}_K$-algebra A_∞ such that A^a_∞ is a perfectoid $\mathcal{O}^a_K$-algebra and such that the reduction $h : A/\pi \to A_\infty/\pi$ is almost injective. Then A is integrally closed in $A[1/\pi]$.*

We prove this Proposition in Appendix A.1. Roughly, the proof is that by the perfectoid assumption, the ring A_∞ is always "almost integrally closed" in its generic fibre. Since h is injective, we may deduce that the same is true for A. By an algebra lemma, we can then remove the "almost" due to the assumption that A is topologically finitely presented.

In turn, in the context of Proposition 4.1.2 it then follows in the limit that the almost equality on perfections is also in fact already an honest equality.

proof of Proposition 4.1.2. To see that $\mathcal{O}_{\mathfrak{X}'(\epsilon)}$ is integrally closed in $\mathcal{O}_{\mathfrak{X}'(\epsilon)}[1/t]$, it suffices to prove that $\mathcal{O}_{\mathfrak{X}(\epsilon)}(U) \subseteq \mathcal{O}^+_{\mathcal{X}(\epsilon)}(U)$ is integrally closed for a basis of affine opens $U \subseteq \mathfrak{X}(\epsilon)$. Instead of working with t, we can more generally work with any pseudo-uniformiser $\pi \in \mathcal{O}_K$. For $1 > \epsilon$ we could take $\pi := t^{1-\epsilon}$, which will be useful later.

Since $\mathfrak{X}(\epsilon)$ is covered by opens of the form $\mathrm{Spf}(S)$ for $S = R\langle X\rangle/(X\mathrm{Ha} - t^\epsilon)$ where $\mathrm{Spf}(R) \subseteq \mathfrak{X}$, it suffices to prove that formal localisations of the form $T := S\langle f^{-1}\rangle$ are integrally closed in their π-adic generic fibre, where $f \in S$ is a non-zero-divisor mod π.

To see this, let $\mathrm{Spf}(T_\infty) \subseteq \mathfrak{X}(\epsilon)^{\mathrm{perf}}$ be the pullback of $\mathrm{Spf}(T)$. We wish to apply Proposition 4.1.4 to the morphism $T \to T_\infty$. To see that the necessary assumptions are satified for the Proposition to apply, we first note that T is indeed of topologically finite presentation over R as is clear from its construction. It is clear that T^a_∞ is a perfectoid $\mathcal{O}^a_K$-algebra. We are therefore left to see that $T/\pi \to T_\infty/\pi$ is almost injective. Since perfection commutes with formal localisation, thus $T_\infty = S^{\mathrm{perf}}\langle f^{-1}\rangle$, this is more precisely the morphism

$$S[f^{-1}]/\pi \to S^{\mathrm{perf}}[f^{-1}]/\pi.$$

Since $R\langle f^{-1}\rangle/\pi = R/\pi[f^{-1}]$ is a flat R/π-algebra, it suffices to see that the map $S/\pi \to S^{\mathrm{perf}}/\pi$ is almost injective. As the latter is the direct limit over Frobenius relative to $\mathcal{O}_K/\pi$, it suffices to see that the relative Frobenius of $S/\pi = R/\pi[X]/(X\mathrm{Ha}-t^\epsilon)$ is almost injective.

But this is easy to see: By an induction argument, we may without loss of generality assume $\pi = t^\epsilon$. By base-change along the map $\mathbb{F}_p[[t]] \to \mathcal{O}_K$ that sends t to π, we may reduce to the Noetherian situation, where $R_{\mathbb{F}_p[[t]]}\langle X\rangle/(X\mathrm{Ha}-t)$ is normal by Lemma 3.1.3. Since the relative Frobenius is finite, this then shows that it is indeed injective mod t.

Therefore Proposition 4.1.4 applies and shows that $T \subseteq T[1/\pi]$ is integrally closed.

This shows that $\mathcal{O}_{\mathfrak{X}(\epsilon)}$ is integrally closed in $\mathcal{O}_{\mathfrak{X}(\epsilon)}[1/t]$. By Lemma A.2.3, it follows that $\mathcal{O}_{\mathfrak{X}(\epsilon)}(U) = \mathcal{O}^+_{\mathcal{X}(\epsilon)}(U)$. This finishes the proof of part 1 for K of characteristic p.

To deduce part 2 from part 1, note that by Lemma A.2.3 we have in fact proved that $\mathcal{O}_{\mathfrak{X}(\epsilon)}(U) = \mathcal{O}_{\mathcal{X}(\epsilon)}(U)^\circ$. Consequently, Lemma A.2.2.1/3 applied to the system

$$S \xrightarrow{F_{\mathrm{rel}}} S \xrightarrow{F_{\mathrm{rel}}} \dots$$

show that also S^{perf} is integrally closed in $S^{\mathrm{perf}}[1/\pi]$, and $S^{\mathrm{perf}} = S^{\mathrm{perf}}[1/\pi]^\circ$. Since $\mathfrak{X}(\epsilon)^{\mathrm{perf}} \to \mathfrak{X}(\epsilon)$ is a homeomorphism of the underlying topological spaces, the basis of affine opens of $\mathfrak{X}(\epsilon)$ we worked with for part 1 also defines a basis of opens of $\mathfrak{X}(\epsilon)^{\mathrm{perf}}$. We deduce that we have an equality of sheaves $\mathcal{O}_{\mathfrak{X}(\epsilon)^{\mathrm{perf}}} = \mathcal{O}^+_{\mathcal{X}(\epsilon)^{\mathrm{perf}}}$ as desired. □

proof of Proposition 4.1.3. Part 1 follows exactly like in Proposition 4.1.2.1: Instead of $\mathfrak{X}(\epsilon)^{\mathrm{perf}}$ we use the integral perfectoid cover $\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}(\epsilon)$. Since the morphism $\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)/p^{1-\epsilon} \to \mathfrak{X}(\epsilon)/p^{1-\epsilon}$ is almost isomorphic to $\mathfrak{X}(\epsilon)^{\mathrm{perf}}/t^{1-\epsilon} \to \mathfrak{X}(\epsilon)/t^{1-\epsilon}$, the above discussion implies that the assumptions of Proposition 4.1.4 are also satisfied in this case.

For the second part, we first recall the isomorphism $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a = \mathfrak{X}(p^{-n}\epsilon)$. By shrinking ϵ, this shows that the statement of part 1 also holds for $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a$ instead of $\mathfrak{X}(\epsilon)$. Consequently, we can again use Lemma A.2.2.1/3 to deduce that $\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}$ is integrally closed in the generic fibre: Here we use that the map $\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}(\epsilon)$ again identifies the underlying topological spaces, since the underlying map can be described as

$$|\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a| = |\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a/p^{1-\epsilon}| = |\mathfrak{X}'(\epsilon)^{\mathrm{perf}}/t^{1-\epsilon}| \xrightarrow{\sim} |\mathfrak{X}'(\epsilon)/t^{1-\epsilon}| = |\mathfrak{X}(\epsilon)/p^{1-\epsilon}| = |\mathfrak{X}(\epsilon)|.$$

Like before this implies $\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a} = \mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}$. This finishes the proof of part 2. □

4.2 Different notions of integral modular forms

For t-adic modular forms, Proposition 4.1.2 now allows us to compare the integral sheaves $\omega^{\kappa,+}$ and $\mathfrak{w}^\kappa$. Reversing this reasoning, we also obtain formal models for the p-adic sheaves.

4.2.1 t-adic formal models

In the following, all analytic sheaves are tacitly pushed forward to the formal models.

Corollary 4.2.1. *Let K be a perfectoid field over $\mathbb{F}_p$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight.*

1. *The natural inclusion $\mathfrak{w}^\kappa \subseteq \omega^{\kappa,+}$ of sheaves on $\mathfrak{X}'(\epsilon)$ is an equality*
$$\mathfrak{w}^\kappa = \omega^{\kappa,+}.$$
2. *The natural inclusion $\mathfrak{w}^{\kappa,\mathrm{perf}} \subseteq \omega^{\kappa,+,\mathrm{perf}}$ of sheaves on $\mathfrak{X}'(\epsilon)^{\mathrm{perf}}$ is an equality*
$$\mathfrak{w}^{\kappa,\mathrm{perf}} = \omega^{\kappa,+,\mathrm{perf}}.$$

Proof. This follow from Proposition 4.1.2 by the same argument as in Lemma 4.1.1. □

Remark 4.2.2. It is clear that for $\epsilon \geq 0$, any integral perfectoid overconvergent modular form has q-expansions with coefficients in $\mathcal{O}_K$. Conversely, using [27], Proposition 6.12, one can show that if all q-expansions of f have coefficients in $\mathcal{O}_K$, then f is integral with respect to $\epsilon = 0$. But this can fail for higher $\epsilon > 0$, as the t-adic modular form Ha^{-1} shows. In particular, this shows that our notion of integrality depends on ϵ.

4.2.2 p-adic formal models and traces

Imitating Corollary 4.2.1, we now also define formal models of the modular sheaves in characteristic 0, in a way such that the analogue of Corollary 4.2.1 is true by construction:

Definition 4.2.3. Let K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$, let $0 \leq \epsilon \leq \epsilon_\kappa$. For $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let $s_n : (\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a, \mathcal{O}^+) \to \mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)$ be the natural morphism. Then we set
$$\mathfrak{w}^\kappa_{\Gamma_0(p^n)} := s_{n*}\omega^{\kappa,+}, \qquad \mathfrak{w}^\kappa := \mathfrak{w}^\kappa_{\Gamma_0(1)}, \qquad \mathfrak{w}^{\kappa,\mathrm{perf}} := \mathfrak{w}^\kappa_{\Gamma_0(p^\infty)}.$$

Proposition 4.2.4. *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.*

1. *The sheaf $\mathfrak{w}^\kappa_{\Gamma_0(p^n)}$ is a line bundle on $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a$, and we have $s_n^*\mathfrak{w}^\kappa_{\Gamma_0(p^n)} = \omega^{\kappa,+}_{\Gamma_0(p^n)}$.*
2. *The sheaf $\mathfrak{w}^{\kappa,\mathrm{perf}}$ is the pullback of $\mathfrak{w}^\kappa_{\Gamma_0(p^n)}$ along $q : \mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a$.*

Proof. Recall that for $n < \infty$, we have $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a = \mathfrak{X}(p^{-n}\epsilon)$. By Proposition 4.1.3, we therefore have $s_{n*}\mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a} = \mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a}$. It therefore suffices to see that $\omega^{\kappa,+}_{\Gamma_0(p^n)}$ has an invertible section already locally on $\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)$. For $n = 0$ this follows from Proposition 2.7.6 since ω is trivial locally on $\mathfrak{X}(\epsilon)$. Since by Proposition 2.4.1, the sheaf $\omega^{\kappa,+}_{\Gamma_0(p^n)}$ is the pullback of $\omega^{\kappa,+}$ along $\lambda : \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}(\epsilon)$, the same section works for $n > 1$.

The same proof works for $s_\infty^*\mathfrak{w}^{\kappa,\mathrm{perf}}$: Here the local section exists by pullback using that $q^*\omega^{\kappa,+} = \omega^{\kappa,+,\mathrm{perf}}$ by Corollary 2.9.1. This proves part 1. Since in particular $\mathfrak{w}^\kappa_{\Gamma_0(p^n)}$ and $\mathfrak{w}^{\kappa,\mathrm{perf}}$ are locally trivialised by the same section, this also proves part 2. □

Corollary 4.2.5. *Let K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight, $0 \leq \epsilon \leq \epsilon_\kappa$. Then the inclusions $M^+_{\kappa,\Gamma_0(p^n)}(\epsilon) \hookrightarrow M^{+,\mathrm{perf}}_\kappa(\epsilon)$ for $n \in \mathbb{Z}_{\geq 0}$ have compatible $\mathcal{O}_K$-linear sections*
$$M^{+,\mathrm{perf}}_\kappa(\epsilon) \to p^{-3\epsilon/p^{n-1}(p-1)} M^+_{\kappa,\Gamma_0(p^n)}(\epsilon).$$

Proof. The inclusion $\mathcal{O}_{\mathfrak{X}(\epsilon)} \to \mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}$ has an $\mathcal{O}_K$-linear normalised Tate trace
$$\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a} \to p^{-3\epsilon p/(p-1)}\mathcal{O}_{\mathfrak{X}(\epsilon)}.$$

This could be seen by replacing $\mathbb{Z}_p^{\mathrm{cyc}}$ with $\mathcal{O}_K$ in [50] Corollaries III.2.22 and III.2.23, which is possible without changing the proofs. We will instead deduce this from a very similar trace computation in Corollary 8.1.5. The compatible Frobenius lifts $\mathfrak{X}(p^{-n}\epsilon) \xrightarrow{\sim} \mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)_a$ and $\mathfrak{X}_{\Gamma_0(p^\infty)}(p^{-n}\epsilon)_a \xrightarrow{\sim} \mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ identify this with the morphism

$$\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a} \to p^{-3\epsilon/p^{n-1}(p-1)}\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^n)}(\epsilon)}.$$

By Proposition 4.2.4.2, upon tensoring with $\mathfrak{w}^\kappa_{\Gamma_0(p^n)}$ this becomes a map

$$q_*\mathfrak{w}^{\kappa,\mathrm{perf}} \to p^{-3\epsilon/p^{n-1}(p-1)}\mathfrak{w}^\kappa_{\Gamma_0(p^n)}.$$

The Lemma follows by taking global sections. □

4.3 Base change

In this section, we discuss base change properties of the spaces of modular forms, and in particular their integral subspaces. This is the analogue in our setting of the classical base change result for modular forms [34], Theorem 1.7.1. The results in this sections all follow from general facts about complexes of complete modules discussed in Appendix A.4.

Throughout, if not explicitly stated otherwise, we fix a perfectoid field K that is either of characteristic p or a field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$, and we fix a pseudo-uniformiser $\pi \in \mathcal{O}_K$.

Lemma 4.3.1. *The $\mathcal{O}_K$-modules $M^+_\kappa(\epsilon)$ and $M^{+,\mathrm{perf}}_\kappa(\epsilon)$ are flat and π-adically complete.*

Proof. These are both special cases of Lemma A.3.8.1. □

Definition 4.3.2. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight. We usually understand the base field of $M^+_\kappa(\epsilon)$ to be the field K implicit in κ. But when L is a perfectoid field extension of K, we can also see κ as a weight valued in L. If the base field is not clear from the context, we distinguish the associated modules of modular forms by denoting them by $M_{\kappa,K}(\epsilon)$ and $M_{\kappa,L}(\epsilon)$ respectively. Similarly for perfectoid and integral modular forms.

Proposition 4.3.3. *Let either $K = \mathbb{F}_p((t))$ and $\kappa = \overline{\kappa}$ be the canonical weight, or let K be a perfectoid field over $\mathbb{F}_p((t))$ or $\mathbb{Q}_p^{\mathrm{cyc}}$ and let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any non-trivial weight. Let $0 \le \epsilon \le \epsilon_\kappa$ be such that $\epsilon \in \log|K|$ and let $L \supseteq K$ be a perfectoid extension. Then*

$$M^+_{\kappa,L}(\epsilon) = M^+_{\kappa,K}(\epsilon)\hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_L \quad \text{and} \quad M^{+,\mathrm{perf}}_{\kappa,L}(\epsilon) = M^{+,\mathrm{perf}}_{\kappa,K}(\epsilon)\hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_L.$$

For now, we only prove this for $K = \mathbb{F}_p((t))$ and postpone the proof in the other cases to §11.3. This is no problem since the other cases will not be applied before section §12.

Proof of Proposition 4.3.3 for $K = \mathbb{F}_p((t))$. Denote by $\mathfrak{w}^\kappa_{\mathcal{O}_L}$ the base-change of $\mathfrak{w}^\kappa$ along $\mathfrak{X}_{\mathcal{O}_L}(\epsilon) \to \mathfrak{X}_{\mathcal{O}_K}(\epsilon)$. Then by Proposition 4.2.1, it suffices to show $\Gamma(\mathfrak{w}^\kappa_{\mathcal{O}_L}) = \Gamma(\mathfrak{w}^\kappa)\hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_L$. But in case that K is discrete, this follows from Lemma A.3.8.2. Similarly for $\mathfrak{w}^{\kappa,\mathrm{perf}}$. □

Corollary 4.3.4. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight and let $0 \le \epsilon \le \epsilon_\kappa$ with $\epsilon \in \log|K|$.*

1. *An element $h \in \Gamma(\mathfrak{w}^\kappa/\pi)$ lifts to an element of $M^+_\kappa(\epsilon) = \Gamma(\mathfrak{w}^\kappa)$ if and only if there is a field extension $L|K$ for which the image $h_{\mathcal{O}_L} \in \Gamma(\mathfrak{w}^\kappa_{\mathcal{O}_L}/\pi)$ lifts to $M^+_{\kappa,\mathcal{O}_L}(\epsilon)$.*
2. *An element $h \in \Gamma(\mathfrak{w}^{\kappa,\mathrm{perf}}/\pi)$ lifts to $M^{+,\mathrm{perf}}_\kappa(\epsilon) = \Gamma(\mathfrak{w}^{\kappa,\mathrm{perf}})$ if and only if there is a field extension $L|K$ for which the image $h_{\mathcal{O}_L} \in \Gamma(\mathfrak{w}^{\kappa,\mathrm{perf}}_{\mathcal{O}_L}/\pi)$ lifts to $M^{+,\mathrm{perf}}_{\kappa,\mathcal{O}_L}(\epsilon)$.*
3. *An element $f \in M^{+,\mathrm{perf}}_\kappa(\epsilon)/\pi$ is contained in $M^+_\kappa(\epsilon)/\pi$ if and only if there is a field extension $L|K$ for which the image $f_{\mathcal{O}_L} \in M^{+,\mathrm{perf}}_{\kappa,\mathcal{O}_L}(\epsilon)/\pi$ is contained in $M^+_{\kappa,\mathcal{O}_L}(\epsilon)/\pi$.*

Proof. Write $b : \mathfrak{X}_{\mathcal{O}_L}(\epsilon) \to \mathfrak{X}(\epsilon)$ for the base-change map. Then the natural morphism $\mathfrak{w}^\kappa \to b_*\mathfrak{w}^\kappa_{\mathcal{O}_L}$ induced a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathfrak{w}^{\kappa})/\pi \arrow[r] \arrow[d] & \Gamma(\mathfrak{w}^{\kappa}/\pi) \arrow[r] \arrow[d] & H^1(\mathfrak{w}^{\kappa})[\pi] \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \Gamma(\mathfrak{w}^{\kappa}_{\mathcal{O}_L})/\pi \arrow[r] & \Gamma(\mathfrak{w}^{\kappa}_{\mathcal{O}_L}/\pi) \arrow[r] & H^1(b_*\mathfrak{w}^{\kappa}_{\mathcal{O}_L})[\pi] \arrow[r] & 0
\end{tikzcd}$$

By Proposition 4.3.3 and its proof, the morphism on the bottom left is simply the morphism on the top left tensored with $\mathcal{O}_L$. Since $\mathcal{O}_L$ is flat, this shows that the vertical morphism g in the diagram is also tensoring with $\mathcal{O}_L$. Since $\mathcal{O}_L/\mathcal{O}_K$ is also $\mathcal{O}_K$-flat, we conclude that g is injective. This shows that the square on the left is a pullback square, as desired.

The proof of the second and third part is the same, replacing $0 \to \Gamma(\mathfrak{w}^{\kappa})/\pi \to \Gamma(\mathfrak{w}^{\kappa}/\pi)$ with $0 \to \Gamma(\mathfrak{w}^{\kappa,\mathrm{perf}})/\pi \to \Gamma(\mathfrak{w}^{\kappa,\mathrm{perf}}/\pi)$ and $0 \to M^+_{\kappa}(\epsilon)/\pi \to M^{+,\mathrm{perf}}_{\kappa}(\epsilon)/\pi$, respectively. $\square$

5 The convergent tilting isomorphism of modular forms

In this section, we prove the first simple version of the tilting isomorphism of modular forms.

Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$ and let t be a pseudo-uniformiser of $K^\flat$ such that $\mathcal{O}_{K^\flat}/t = \mathcal{O}_K/p$. In the last sections, we have developed in parallel the theory of p-adic modular forms over K and t-adic modular forms over $K^\flat$ via analytic and perfectoid modular curves. It is apparent from the definitions that there is a close analogy between the two geometric setups as well as the two definitions of modular forms. The goal of this section is to make this precise using the language of perfectoid space.

We start by recalling Scholze's tilting isomorphism of modular curves at level $\Gamma_0(p^\infty)$, and then discuss an extension to the modular curve of level $\Gamma_1(p^\infty)$. This will immediately give rise to a first weak version of the tilting isomorphism for perfectoid modular forms, restricted to certain families of weights. Using classical methods like multiplication by Eisenstein series, we can strengthen this in the case of $\epsilon = 0$ of convergent modular forms.

We note that the proof of the overconvergent version will ultimately not need the convergent tilting isomorphism as an input. The purpose of the discussion in this section is therefore mainly to illustrate the machinery behind the tilting isomorphism, and to motivate why the additional ingredients developed in the following sections are necessary.

5.1 The tilting isomorphism of modular curves

In this section, we recall Scholze's tilting isomorphism of modular curves, and discuss its extension to the perfectoid modular curve $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$. We closely follow [50], §III.2, except for the minor difference that we work over perfectoid extensions of $\mathbb{Q}_p^{\mathrm{cyc}}$. We recall that we denote by $\mathcal{X}$ the compactified modular curve, whereas Scholze denotes this by $\mathcal{X}^*$.

Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$ and let $0 \leq \epsilon \in \log|K|$ with $\epsilon < 1/2$. Recall that we have defined the ϵ-overconvergent neighbourhood $\mathcal{X}(\epsilon)$ of the locus of ordinary reduction of the modular curve over K, and analogously $\mathcal{X}'(\epsilon)$ over $K^\flat$. We have already discussed in §3 the canonical formal model $\mathfrak{X}'(\epsilon)$ of this space and discussed that the relative Frobenius is of the form $\phi : \mathfrak{X}'(p^{-1}\epsilon) \to \mathfrak{X}'(\epsilon)$.

In [50], III.2.2, Scholze constructs the perfectoid space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$. He does so by considering a canonical formal model $\mathfrak{X}(\epsilon)$ of $\mathcal{X}(\epsilon)$ which is defined exactly like in (12). In particular, this local definition shows that we have a canonical identification

$$\mathfrak{X}(\epsilon)/p^{1-\epsilon} = \mathfrak{X}'(\epsilon)/t^{1-\epsilon}. \tag{19}$$

Via the Atkin–Lehner isomorphism from §2.4, we may regard $\mathfrak{X}(p^{-n}\epsilon)$ as a formal model for $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)$. By Lemma 2.1.2, the map $\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}(\epsilon)$ then admits a formal model

$\phi^n : \mathfrak{X}(p^{-n}\epsilon) \to \mathfrak{X}(\epsilon)$, defined using canonical subgroups, that reduces to the relative Frobenius mod $p^{1-\epsilon}$. In particular, this means that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}(p^{-n}\epsilon)/p^{1-\epsilon} & \xrightarrow{\phi^n} & \mathfrak{X}(\epsilon)/p^{1-\epsilon} \\ \| & & \| \\ \mathfrak{X}'(p^{-n}\epsilon)/t^{1-\epsilon} & \xrightarrow{\phi^n} & \mathfrak{X}'(\epsilon)/t^{1-\epsilon}. \end{array}$$

In the limit over n, this shows that the formal scheme $\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a := \varprojlim_\phi \mathfrak{X}(p^{-n}\epsilon)$ satisfies

$$\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a/p^{1-\epsilon} = \mathfrak{X}'(\epsilon)^{\mathrm{perf}}/t^{1-\epsilon}.$$

This is how Scholze proves that the adic generic fibre $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a := (\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a)^{\mathrm{ad}}_\eta$ is perfectoid. Via the tilting equivalence, it also immediately gives an identification of the tilt:

Theorem 5.1.1 ([50], Corollary III.2.19)**.** *There is a natural isomorphism*

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a = \mathcal{X}'(\epsilon)^{\mathrm{perf}}.$$

The significance of this result to modular forms is that this identifies the base spaces of the bundles of p-adic perfectoid and t-adic perfectoid modular forms, respectively. From this perspective, the following extension of Theorem 5.1.1 says that one is moreover able to also identify the domains of definition of perfectoid modular forms:

Theorem 5.1.2 ([27], Theorem 5.6)**.** *There is a natural $\mathbb{Z}_p^\times$-equivariant isomorphism*

$$\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a = \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}},$$

compatible with the isomorphism from Theorem 5.1.1. Over any cusp of $\mathcal{X}(\epsilon)$, the tilt of the cusp morphism $\underline{\mathbb{Z}_p^\times} \times D_\infty \hookrightarrow \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ fits into a commutative diagram of isomorphisms

$$\begin{array}{ccc} (\underline{\mathbb{Z}_p^\times} \times D_\infty)^\flat & \hookrightarrow & \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \\ \| & & \| \\ \underline{\mathbb{Z}_p^\times} \times D'_\infty & \hookrightarrow & \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}. \end{array}$$

Let us say a few words about the proof: A version of this result is proved for Siegel varieties of dimension ≥ 2 in [50], III.2.5 as an intermediate step in the construction of the Siegel variety of level $\Gamma(p^\infty)$. More precisely, it is required to take care of ramification at the boundary in the tower from level $\Gamma_0(p^\infty)$ to level $\Gamma_1(p^\infty)$. For the proof, Scholze uses heavy technical machinery like his Hebbarkeitssatz and Hartog's extension principle, which does not carry over to the case of dimension 1, in which the tower is much easier to understand since it is unramified. The proof of Lemma III.2.29, however, does carry over, which means that Scholze proves the above tilting identification over the good reduction locus. In [27], we complete the proof in the case of dimension 1 by carrying out a systematic analysis of what happens at the cusps. More precisely, it is easy to first extend to the result to the open modular curve. It is then precisely the statement about Tate parameter spaces in the Theorem which extends the isomorphism to the compactified spaces.

One could instead also deduce the isomorphism of Theorem 5.1.2 from the perfected Igusa formal scheme over $\mathbb{Z}_p[[T]]$ constructed in §6 [3]. A third strategy would be to simply embed the modular curve into the Siegel 3-fold by sending $E \mapsto E \times E$ and deduce the statement from the case of higher dimensional Siegel varieties (we learnt this trick from [50], proof of Theorem III.2.15). However, neither of these strategies would tell us anything precise about what happens on Tate parameter spaces at the boundary, i.e. the second part of Theorem 5.1.2. Our specific interest in this part stems from the fact that it will enable us to keep track of what happens on q-expansions in the tilting isomorphism of modular forms.

5.2 The tilting isomorphism of perfectoid modular forms

Theorem 5.1.2 is the basis of the tilting isomorphism of modular forms. While we need more ingredients to prove the result in full generality, we are ready to give a first simple version.

Definition 5.2.1. Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be any weight. For any $n \in \mathbb{Z}$, composing $\kappa^\flat$ with $F_{\mathrm{abs}}^{-n} : x \mapsto x^{1/p^n}$ defines another weight, which to keep notation light we denote by

$$\kappa_n^\flat : \mathbb{Z}_p^\times \xrightarrow{\kappa} \mathcal{O}_{K^\flat}^\times \xrightarrow{F_{\mathrm{abs}}^{-n}} \mathcal{O}_{K^\flat}^\times.$$

Composing further with the continuous homomorphism $\sharp : K^{\flat\times} \to K^\times$, we obtain a weight

$$\kappa_n^{\flat\sharp} : \mathbb{Z}_p^\times \xrightarrow{\kappa} \mathcal{O}_{K^\flat}^\times \xrightarrow{F_{\mathrm{abs}}^{-n}} \mathcal{O}_{K^\flat}^\times \xrightarrow{\sharp} \mathcal{O}_{K^\flat}^\times.$$

Theorem 5.2.2. *Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be any weight. Let $\epsilon \in \log|K|$ be such that $0 \le \epsilon \le \epsilon_{\kappa^\flat}$.*

1. *The isomorphism $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a^\flat = \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ induces an isomorphism of $\mathcal{O}_{K^\flat}$-modules*

$$M_{\kappa^\flat}^{\mathrm{perf},+}(\epsilon) = \varprojlim_{f \mapsto f^p} M_{\kappa_n^{\flat\sharp}}^{\mathrm{perf},+}(\epsilon)/p,$$

which on q-expansions is defined by $\mathcal{O}_{K^\flat}[[q^{1/p^\infty}]] = \varprojlim_{x \mapsto x^p} \mathcal{O}_K/p[[q^{1/p^\infty}]]$.

2. *Sending $f \mapsto ((f^{1/p^n})^\sharp)_{n \in \mathbb{Z}_{\ge 0}}$ moreover defines an isomorphism of $K^\flat$-vector spaces*

$$M_{\kappa^\flat}^{\mathrm{perf}}(\epsilon) = \varprojlim_{f \mapsto f^p} M_{\kappa_n^{\flat\sharp}}^{\mathrm{perf}}(\epsilon),$$

which on q-expansions is defined by $K^\flat[[q^{1/p^\infty}]] = \varprojlim_{x \mapsto x^p} K[[q^{1/p^\infty}]]$.

Remark 5.2.3. For the case of the trivial weight $\kappa = 1$, these recover the usual tilting isomorphisms $R^{\flat\circ} = \varprojlim_{x \mapsto x^p} R^{\flat\circ}/p$ and $R^\flat = \varprojlim_{x \mapsto x^p} R$. For general κ, we therefore think of the Theorem as an analogue of these isomorphisms for perfectoid modular forms. This is our first motivation for calling these modular forms "perfectoid".

Proof. To simplify notation, let us write $\kappa_n := \kappa_n^{\flat\sharp}$. We first note that the right hand sides of the isomorphisms are well-defined: Indeed, since we have $(\kappa_{n+1})^p = \kappa_n$, sending $f \mapsto f^p$ defines a map $M_{\kappa_{n+1}}^{+,\mathrm{perf}} \to M_{\kappa_n}^{+,\mathrm{perf}}$ which becomes $\mathcal{O}_K/p$-linear after reduction mod p.

Let us denote $A := \mathcal{O}(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)$ and $A^+ := \mathcal{O}^+(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)$. Then A is a perfectoid K-algebra: The space $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ is affinoid perfectoid since it is pro-étale over the perfectoid space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ which is affinoid perfectoid by [50], Corollary III.2.20. By Theorem 5.1.2, the same is true for $\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ and we have $A^\flat = \mathcal{O}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})$. Let $A^{\flat,+} = \mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})$. By the tilting equivalence, we then have isomorphisms

$$(i)\ \ A^{\flat,+} = \varprojlim_{x \mapsto x^p} A^+/p, \qquad (ii)\ \ A^\flat = \varprojlim_{x \mapsto x^p} A.$$

Theorem 5.1.2 moreover says that these are equivariant for the $\mathbb{Z}_p^\times$-action on both sides.

Let now $(f_i)_{i \in \mathbb{Z}_{\ge 0}}$ be a compatible system of modular forms as in part 2. By the $K^\flat = \varprojlim_{x \mapsto x^p} K$-linear isomorphism (ii), this defines an element $f^\flat \in A^\flat$. By $\mathbb{Z}_p^\times$-equivariance,

$$\gamma^* f^\flat = \gamma^*(f_i) = (\gamma^* f_i) = (\kappa_n(\gamma)^{-1} f_i) = \kappa^\flat(\gamma)^{-1} f^\flat \quad \text{for any } \gamma \in \mathbb{Z}_p^\times.$$

This shows that $f^\flat \in M_{\kappa^\flat}^{\mathrm{perf}}(\epsilon)$. We have thus defined a map $\varprojlim_{f \mapsto f^p} M_{\kappa_n}^{\mathrm{perf}}(\epsilon) \to M_\kappa^{\mathrm{perf}}(\epsilon)$. Using the isomorphism (i) instead, we similarly obtain $\varprojlim_{f \mapsto f^p} M_{\kappa_n^\sharp}^{\mathrm{perf},+}(\epsilon)/p \to M_\kappa^{\mathrm{perf},+}(\epsilon)$.

Conversely, given a modular form $f \in M^{\mathrm{perf}}_{\kappa^\flat}(\epsilon)$, one sees by reversing the above calculation that $f^\sharp$ is a modular form of weight $\kappa^{\flat\sharp} = \kappa_0$. Since f^{1/p^n} is clearly a modular form of weight $\kappa^\flat_n$, this shows that $((f^{1/p^n})^\sharp)_{n\in\mathbb{Z}_{\geq 0}}$ defines an element in $\varprojlim_{f\mapsto f^p} M^{\mathrm{perf}}_{\kappa_n}(\epsilon)$. Since $f^\sharp$ is integral if and only if f is, the same construction gives the inverse map in part 2.

The statement about q-expansions follows from the second part of Theorem 5.1.2. $\square$

Remark 5.2.4. Theorem 5.2.2 requires the p-adic weights to satisfy $\kappa^p_{n+1} = \kappa_n$. Since this limits our choices of κ_n, it is desirable to replace this by a weaker condition: For example, we are particularly interested in classical weights κ_n since for these we have a classical subspace of modular forms that we understand well. But for general perfectoid fields K, it is not clear that a p-th root κ_{n+1} of such a κ_n exists, and even if it does, κ_{n+1} might not itself be classical (take for example $k = 2$ for $p > 2$). We would therefore like to weaken the condition to be $\kappa^p_{n+1} \equiv \kappa_n \bmod p$. For example, we note that for any classical weight κ_n, there are infinitely many classical weights κ_{n+1} for which this weaker condition holds. Nevertheless, any sequence $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K)_{n\in\mathbb{N}}$ satisfying $\kappa^p_{n+1} \equiv \kappa_n \bmod p$ still defines a weight $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}^\times_{K^\flat}$ via $\mathcal{O}^\times_{K^\flat} = \varprojlim_{x\mapsto x^p}(\mathcal{O}_K/p)^\times$, and the proof of Theorem 5.2.2 immediately generalises to show that any sequence of perfectoid modular forms $f_n \in M^{\mathrm{perf},+}_{\kappa_n}(\epsilon)$ with $f^p_{n+1} \equiv f_n \bmod p$ defines a t-adic perfectoid modular form $f^\flat \in M^{\mathrm{perf},+}_{\kappa^\flat}(\epsilon)$. The problem is thus to show that one can go into the other direction, i.e. assign to a perfectoid modular form of weight $\kappa^\flat$ a compatible sequence of perfectoid modular forms of weight κ_n.

Our overarching goal for the next chapters will be to improve Theorem 5.2.2.1 in two ways: First, we would like to have greater flexibility in the choice of weights on the right hand side of the tilting isomorphism, in the sense of Remark 5.2.4. The second goal is to obtain a version for true t-adic modular forms that is also Hecke-equivariant.

5.3 The convergent tilting isomorphism

The result of the last section is already enough to deduce a tilting isomorphism for true t-adic modular forms in the case of $\epsilon = 0$. We shall refer to these forms as convergent modular forms, as opposed to overconvergent modular forms.

The first characteristic feature of the convergent case is that "everything can be detected from q-expansions": For example, for K a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$, let $c_1, \dots, c_n$ be the cusps of $\mathcal{X}(\epsilon)$, then by [27], Proposition 6.12, the injection $M^{+,\mathrm{perf}}_\kappa(0) \hookrightarrow \prod_{i=1}^n \mathcal{O}_K[[q^{1/p^\infty}]]$ that sends f to its q-expansions at $c_1, \dots, c_n$ remains injective after reducing mod p. As a consequence, the classical theory of the Eisenstein family applies in its full force and lets us move the p-adic weight κ as follows:

Lemma 5.3.1. *Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$.*

1. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight with $\kappa \equiv 1 \bmod p$. Then the Eisenstein series E_κ is a modular form in $M^+_\kappa(0)$ that satisfies $E_\kappa \equiv 1 \bmod p$ inside $\mathcal{O}_K[[q]]$.*
2. *Let $\kappa_1, \kappa_2 : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be two weights with $\kappa_1 \equiv \kappa_2 \bmod p$. Let $\kappa := \kappa_2 \cdot \kappa_1^{-1}$ be their ratio. Then for any $n \in \mathbb{Z}_{\geq 0}$, multiplication with E_κ induces an isomorphism*

$$E_\kappa\cdot : M^+_{\kappa_1,\Gamma_0(p^n)}(0)/p \xrightarrow{\sim} M^+_{\kappa_2,\Gamma_0(p^n)}(0)/p$$

which is given by the identity on q-expansions.

Proof. 1. This is well-known, see for instance [18], §2.2.

2. This follows from 1: It is clear from the definitions that multiplication by E_κ induces a homomorphism of $\mathcal{O}_K$-modules $M^+_{\kappa_1,\Gamma_0(p^n)}(\epsilon) \to M^+_{\kappa_2,\Gamma_0(p^n)}(\epsilon)$. After reducing mod

p, this is the identity on q-expansions since we have $E_\kappa \equiv 1 \bmod p$. In particular, it is injective. Switching the roles of κ_2 and κ_1 shows that it is in fact an isomorphism. $\square$

As an additional simplification, recall from Corollary 3.6.8 that for $\epsilon = 0$, the trace map

$$\mathrm{tr} : M^{+,\mathrm{perf}}(\epsilon) \to M^+(\epsilon)$$

is in fact integral, in contrast to the case of $\epsilon > 0$. On q-expansions, this map can be described as "forgetting coefficients" by sending $\sum_{m\in\mathbb{Z}[1/p]_{\geq 0}} a_m q^m \mapsto \sum_{m\in\mathbb{Z}_{\geq 0}} a_m q^m$.

Combining these three observations, we can deduce from Theorem 5.2.2 a convergent tilting isomorphism. Here the two goals noted at the very end of §5.2 are achieved as follows: We get flexibility of weights on the right hand side in the sense of Remark 5.2.4 by multiplying with Eisenstein series as in Lemma 5.3.1. We then descend to true p-adic modular forms by applying traces, or equivalently, the q-expansion principles Proposition 2.3.4 and 3.5.4.

Definition 5.3.2. For any power series $f := \sum a_m q^m \in \mathcal{O}_K[[q^{1/p^\infty}]]$, we denote by $f^{(p)} \in \mathcal{O}_K[[q^{1/p^\infty}]]$ the power series defined by $f^{(p)} := \sum a_m^p q^m$.

Theorem 5.3.3. *Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ be a family of weights such that $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ for all $n \in \mathbb{Z}_{\geq 0}$. Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be the corresponding weight under $\mathcal{O}_{K^\flat}^\times = \varprojlim_{x\mapsto x^p}(\mathcal{O}_K/p)^\times$. Then the isomorphism $\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a^\flat = \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}$ induces a Hecke-equivariant isomorphism of $\mathcal{O}_{K^\flat}$-modules*

$$M_{\kappa^\flat}^+(0) = \varprojlim_{f\mapsto f^p} M_{\kappa_n,\Gamma_0(p^n)}^+(0)/p = \varprojlim_{f\mapsto f^{(p)}} M_{\kappa_n}^+(0)/p.$$

This is compatible with q-expansions: Given a sequence $(f_n)_{n\in\mathbb{Z}_{\geq 0}}$ on the right hand side with q-expansions $f_n = \sum a_{n,m} q^m$, the q-expansion of the corresponding modular form $f^\flat \in M_{\kappa^\flat}^+(0)$ is given by $f^\flat = \sum a_m^\flat q^m$ where $a_m^\flat = (a_{n,m})_{n\in\mathbb{Z}_{\geq 0}}$ under $\mathcal{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p$.

Proof. We sketch the argument to illustrate the ingredients going into the proof – we will later give an independent proof of a stronger Theorem in much more detail.

Let $\kappa_n^{\flat\sharp} : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be defined like in Definition 5.2.1. The isomorphism $\mathcal{O}_{K^\flat}[[q^{1/p^\infty}]] = \varprojlim \mathcal{O}_K/p[[q^{1/p^\infty}]]$ identifies the subspaces $\mathcal{O}_{K^\flat}[[q]] = \varprojlim_n \mathcal{O}_K/p[[q^{1/p^n}]]$. Consequently, the isomorphism $M_{\kappa^\flat}^{+,\mathrm{perf}}(0) = \varprojlim_{x\mapsto x^p} M_{\kappa_n^{\flat\sharp}}^{+,\mathrm{perf}}(0)/p$ from Theorem 5.2.2.1 commutes with the trace maps on both sides, where in the n-th member on the right we take the trace $\mathrm{tr}_n : M_{\kappa^\flat}^{+,\mathrm{perf}}(0) \to M_{\kappa^{\flat\sharp},\Gamma_0(p^n)}^+(0)$. We therefore get an isomorphism

$$M_\kappa^+(0) = \varprojlim_{x\mapsto x^p} M_{\kappa_n^{\flat\sharp},\Gamma_0(p^n)}^+(0)/p.$$

By definition of $\kappa^\flat$, we have $\kappa_n^{\flat\sharp} \equiv \kappa_n \bmod 1$. Consequently, Lemma 5.3.1.2 tells us that

$$M_{\kappa_n^{\flat\sharp},\Gamma_0(p^n)}^+(0)/p = M_{\kappa_n,\Gamma_0(p^n)}^+(0)/p.$$

The second isomorphism of the Theorem comes from the Atkin–Lehner isomorphisms AL^n, which on q-expansions sends $q^{1/p^n} \mapsto q$. It remains to prove Hecke-equivariance. This can be checked using the explicit description of Hecke operators on q-expansions. $\square$

A closer look at the statement of the convergent tilting isomorphism, Theorem 5.3.3, deduces from this the analogue of Serre's point of view on p-adic modular forms mentioned in the introduction: The analogy is that while Serre in [54] studies p-adic convergence of q-expansions, that is he looks at $\mathcal{O}_K[[q]] = \varprojlim_n \mathcal{O}_K/p^n[[q]]$, we study t-adic convergence of q-expansions, that is we look at $\mathcal{O}_{K^\flat}[[q]] = \varprojlim_{f\mapsto f^{(p)}} \mathcal{O}_K/p[[q]]$. In other words, we use

$$\mathcal{O}_K/p[[q]] \cong \mathcal{O}_{K^\flat}/t[[q]] \xrightarrow{\sim} \mathcal{O}_{K^\flat}/t^{p^n}[[q]], \quad f = \sum a_m q^m \mapsto f^{(p^n)} := \sum a_m^{p^n} q^m. \qquad (20)$$

This gives Corollary 1.2.4 mentioned in the introduction, namely the exact analogue of Katz' reinterpretation of Serre's definition of p-adic modular forms, [34], Theorem 4.5.1.

Remark 5.3.4. Recall that for classical modular forms, we have the K-algebra $\oplus_{k\geq 0}M_k^{\mathrm{cl}}(K)$ of modular forms which gives rise to a canonical isomorphism $X_K = \mathrm{Proj}(\oplus_{k\geq 0}M_k^{\mathrm{cl}}(K))$.

Motivated by [35], Theorem 2.1, one may define "the ring of p-adic modular forms" to be $\mathcal{O}(\mathcal{X}_{\mathrm{Ig}(p^\infty)}(0))$ where $\mathcal{X}_{\mathrm{Ig}(p^\infty)}(0) \to \mathcal{X}(0)$ is the pro-étale $\mathbb{Z}_p^\times$-torsor which relatively parametrises trivialisations $\mu_{p^\infty} \xrightarrow{\sim} C_\infty$ of the canonical p-divisible subgroup of the universal semi-abelian scheme over $\mathcal{X}(0)$. One can show that this is a sousperfectoid adic space.

There is a canonical pro-étale projection map

$$\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a \to \mathcal{X}_{\mathrm{Ig}(p^\infty)}(0)$$

which one may regard as being an analogue of Lemma 2.7.5 for Katz' modular forms. This map is defined as follows: Away from the cusps, the dual of the universal anticanonical subgroup $D = (D_n)_{n\in\mathbb{N}}$ of the universal abelian scheme over $\mathcal{X}_{\Gamma_0(p^\infty)}(0)_a$ can be identified with C_∞ via the collection over all n of natural isomorphisms $C_n(E) \xrightarrow{\sim} C_n(E/D_n) \xrightarrow{\sim} D_n^\vee$ where the first map is the canonical lift of the n-iteration of Verschiebung. Using formal models, one can show that the natural map $\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a \to \mathcal{X}_{\mathrm{Ig}(p^\infty)}(0) \times_{\mathcal{X}(0)} \mathcal{X}_{\Gamma_0(p^\infty)}(0) \to \mathcal{X}(0)$ is an isomorphism (and one can use this to verify the sousperfectoidness statement).

Together with the definition of perfectoid p-adic modular forms, this motivates to regard the algebra $\mathcal{O}(\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a)$ as "the ring of all p-adic perfectoid modular forms".

By analogy, "the ring of all t-adic modular forms" should be $\mathcal{O}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0))$ and "the ring of all t-adic modular forms" should be $\mathcal{O}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}})$.

In this light, we may regard the isomorphism $\mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}}) = \varprojlim \mathcal{O}^+(\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a)/p$ Theorem 5.1.2 as being a tilting isomorphism on the level of "rings of perfectoid modular forms", and we could thus regard Theorem 5.2.2 as being a refinement.

One interesting aspect of this perspective is that as shown by Howe [29], there is a natural Serre–Tate-deformation action by the group $\mathcal{O}_K^{\circ\circ}$ on these rings $\mathcal{O}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(0))$: After choosing an isomorphism $\mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$ over $\mathbb{Q}_p^{\mathrm{cyc}}$, there is a natural morphism $M_{\mathrm{bigIgusa}} \to \mathcal{O}(\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a)$ where M_{bigIgusa} is the base change to K of the adic generic fibre of Howe's big Igusa scheme, and the $\mathcal{O}_{K^\flat}^{\circ\circ}$-action on M_{bigIgusa} descends to a $\mathcal{O}_K^{\circ\circ}$-action on $\mathcal{O}(\mathcal{X}_{\Gamma_1(p^\infty)}(0)_a)$.

An analogous action also exists on $\mathcal{O}(\mathcal{X}_{\mathrm{Ig}(p^\infty)}(0)_a)$, and one can show that these actions can be identified via tilting, so that the above tilting isomorphism is equivariant with respect to Howe's deformation action. However, while the tilting isomorphism overconverges, the action by $\mathcal{O}_K^{\circ\circ}$ does not.

5.4 An example: Tilting Eisenstein series

As an example, we shall now discuss an interesting explicit example of the convergent tilting isomorphism Theorem 5.3.3, namely the p-adic Eisenstein family.

Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight with $\kappa(\mu_{p-1}(\mathbb{Z}_p)) = 1$. Recall from [18], §2.1 that the (p-depleted) p-adic Eisenstein series of weight κ is defined as follows: Denote by ζ^* the p-adic zeta-function of [19] B.1, this is a pseudo-measure on $\mathbb{Z}_p^\times$ with a pole at $\kappa = 1$. More precisely, under the usual identification of the Iwasawa algebra of $1 + q\mathbb{Z}_p$ with $\mathbb{Z}_p[[T]]$, we may regard ζ^* as an element of the fraction field of $\mathbb{Z}_p[[T]]$. By a Theorem of Iwasawa, [54], Théorème 16, ζ^* is then of the form

$$\zeta^* = \frac{2}{T}\cdot g, \qquad \text{where } g \in \mathbb{Z}_p[[T]]^\times. \tag{21}$$

If $\kappa \neq 1$, we can thus evaluate ζ^* at κ to get an element $\zeta^*(\kappa) \in K^\times$, and equation (21) shows $2/\zeta^*(\kappa) \in \mathcal{O}_K$. If $\kappa = (\chi, k)$ is classical, then $\zeta^*(\kappa) = L_p(\chi, 1-k)$ where L_p is the

Kubota–Leopoldt p-adic L-function. For $\kappa = 1$, i.e. $T_\kappa = 0$, we may simply set $2/\zeta^*(\kappa) := 0$.

Definition 5.4.1. The Eisenstein series $E_\kappa \in \mathcal{O}_K[[q]]$ of weight κ is defined by

$$E_\kappa = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \sigma_\kappa^*(n) q^n, \quad \text{where } \sigma_\kappa^*(n) = \sum_{\substack{d|n \\ (d,p)=1}} \kappa(d) d^{-1}.$$

As explained in [18], §2, especially Proposition 2.3.1, this is the q-expansion of a unique overconvergent p-adic modular forms of weight κ and tame level 1 in the sense of Katz, and thus also an overconvergent modular forms of weight κ and tame level N in our language.

Following [58], §2.1, there is also a t-adic Eisenstein series $E_{\kappa^\flat}$ for any non-trivial weight $\kappa^\flat : \mathbb{Z}_p^\times \to K^{\flat\times}$, which can be described as follows: Let $\zeta^*(\kappa^\flat)$ be the image of ζ^* under

$$\frac{1}{T}\mathbb{Z}_p[[T]] \to K^\flat, \quad T \mapsto T_{\kappa^\flat},$$

then $\frac{\zeta^*(\kappa^\flat)}{2} \in K^{\flat\times}$ and $\frac{2}{\zeta^*(\kappa^\flat)} \in \mathcal{O}_K$ by equation (21). We then define $E_{\kappa^\flat} \in \mathcal{O}_{K^\flat}[[q]]$ by

$$E_{\kappa^\flat} = 1 + \frac{2}{\zeta^*(\kappa^\flat)} \sum_{n=1}^{\infty} \sigma_{\kappa^\flat}^*(n) q^n.$$

Lemma 5.4.2. *$E_{\kappa^\flat}$ is the q-expansion of a unique t-adic overconvergent modular form of weight $\kappa^\flat$ and radius of overconvergence $\epsilon_{\kappa^\flat}$ as defined in Definition 3.3.6.*

Proof. For $\kappa^\flat = \overline{\kappa}$ the canonical weight, this follows from [58], Proposition 2.1 by specialisation at the boundary. The general case follows as usual by base-change along the map $\iota : \mathbb{F}_p[[t]] \to \mathcal{O}_K$, $t \mapsto T_{\kappa^\flat}$ using Proposition 4.3.3. □

Proposition 5.4.3. *Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ be any sequence of weights satisfying $\kappa_{n+1}^p \equiv \kappa_n \bmod p$, then inside $\mathcal{O}_K[[q]]$,*

$$(E_{\kappa_{n+1}})^p \equiv E_{\kappa_n} \bmod p.$$

If moreover $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ is the corresponding t-adic weight, then $E_{\kappa^\flat}$ corresponds to the sequence $(E_{\kappa_n})_{n\in\mathbb{Z}_{\geq 0}}$ under the isomorphism $M_{\kappa^\flat}^+(0) = \varprojlim M_{\kappa^n}^+(0)/p$ of Theorem 5.3.3.

Remark 5.4.4. The same works more generally for any Hida family with q-expansion in $\mathbb{Z}_p[[T]][[q]]$, with precisely the same proof. Since the proof is an elementary computation, we do not think of this tilting perspective on the Eisenstein family, or on Hida families, as being in any way deep. Instead, we think of these families as being an illustrating example, or a "sanity check", for the tilting equivalence. Moreover, they provide interesting evidence for Conjecture 13.2.1 below, which suggests that there is a tilting correspondence of eigenforms.

The proof of the Proposition is based on the following very simple observation:

Lemma 5.4.5. *Let $h \in \mathbb{Z}_p[[T]]$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of topologically nilpotent elements in $\mathcal{O}_K$ such that $x_{n+1}^p \equiv x_n \bmod p$. Let $x^\flat$ be the image of this sequence under $\varprojlim_{x\mapsto x^p} \mathcal{O}_K/p = \mathcal{O}_{K^\flat}$. Then $h(x_{n+1})^p \equiv h(x_n) \bmod p$ and $(h(x_n))_{n\in\mathbb{N}}$ corresponds to $h(x^\flat)$.*

Proof. Recall that for any $c \in \mathbb{Z}_p$,

$$c^p \equiv c \bmod p. \tag{22}$$

Let $h = \sum a_m T^m$, then by applying (22) to the coefficients of h we see that inside $\mathcal{O}_K$:

$$h(x_{n+1})^p \equiv \sum a_m^p x_{n+1}^{pm} \equiv \sum a_m x_n^m = h(x_n) \bmod p.$$

To see that $(h(x_n))_{n\in\mathbb{N}}$ is sent to $h(x^\flat)$ by $\varprojlim_{x\mapsto x^p} \mathcal{O}_K/p \xrightarrow{\sim} \mathcal{O}_{K^\flat}$, we note that since $\varprojlim_{x\mapsto x^p} \mathbb{Z}_p/p = \mathbb{F}_p$, this isomorphism sends $(c)_{n\in\mathbb{N}}$ for any $c \in \mathbb{Z}_p$ to $(\bar{c})$ where $\bar{c} \in \mathbb{F}_p$ is the reduction. It moreover sends $(x_n)_{n\in\mathbb{N}}$ to $x^\flat$ by definition. By linearity, we conclude that it sends $(h(x_n))_{n\in\mathbb{N}} = (\sum_m a_m x_n^m)_{n\in\mathbb{N}} \mapsto (\sum_m \bar{a}_m x^{\flat m})_{n\in\mathbb{N}} = h(x^\flat)$, as desired. □

Proof of Proposition 5.4.3. By the part about q-expansions in Theorem 5.3.3, it suffices to prove both statements coefficient-by-coefficient. The statement then follows by regarding the Eisenstein family E_κ as a power series $\mathbb{Z}_p[[T]][[q]]$ and applying Lemma 5.4.5 to each q-expansion coefficient separately.

We shall now spell this out explicitly in order to give a hands-on example of the tilting isomorphism at work: We first note that by applying equation (22) to d^{-1} for any $d \in \mathbb{N}$ with $(p, d) = 1$, we see that the assumption $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ implies

$$\left(\sigma^*_{\kappa_{n+1}}(n)\right)^p \equiv \sum_{\substack{d|n\\(d,p)=1}} \kappa_{n+1}^p(d)d^{-p} \equiv \sum_{\substack{d|n\\(d,p)=1}} \kappa_n(d)d^{-1} = \sigma^*_{\kappa_n}(n) \bmod p.$$

To prove that $E^p_{\kappa_{n+1}} \equiv E_{\kappa_n} \bmod p$, it therefore remains to see that

$$\left(\frac{2}{\zeta^*(\kappa_{n+1})}\right)^p \equiv \frac{2}{\zeta^*(\kappa_n)} \bmod p,$$

where we recall that both sides are elements in $\mathcal{O}_K$. But this congruence follows from equation (21), which says that there is a power series $h = Tg^{-1} \in \mathbb{Z}_p[[T]]$ with coefficients in $\mathbb{Z}_p$ such that we have $\frac{2}{\zeta^*(\kappa)} = h(T_\kappa)$ for any weight κ (p-adic as well as t-adic). Since $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ implies $T^p_{\kappa_{n+1}} \equiv T_{\kappa_n} \bmod p$, Lemma 5.4.5 now shows that $h(T_{\kappa_{n+1}})^p \equiv h(T_{\kappa_n}) \bmod p$ and that $(h(T_{\kappa_n}))_{n\in\mathbb{N}} = (\frac{2}{\zeta^*(\kappa_{n+1})})_{n\in\mathbb{N}}$ corresponds to $h(T_{\kappa^\flat})$, which equals $\frac{2}{\zeta^*(\kappa^\flat)}$ by definition. This gives the desired tilting identification on coefficients. □

6 Modular curves over the perfected weight space

Motivated by the convergent tilting equivalence, we begin in this section a new line of investigation into t-adic modular forms: Building on the work of Andreatta–Iovita–Pilloni, the overarching goal of the next sections is to systematically discuss integral families of modular forms over a perfected version of weight space that represents p-power compatible sequences of weights as they appear in the tilting equivalence.

The main goal which motivates our constructions is to relate the Witt vector lift of the perfect t-adic formal scheme $\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ to the integral family of Igusa towers over the perfected weight space, and to use this to canonically extend perfectoid t-adic modular forms to families of perfectoid modular forms over a large weight space annulus.

In order to deduce a version of this lifting for true t-adic modular forms, we need a version of the Tate traces of [50] and [3] in this setting. In order to minimise the failure of these traces to be integral, we work over large annuli that extend from the boundary far towards the centre of weight space. On the way, we can use these traces to prove sousperfectoidness results, which we in turn need for the lift of true t-adic modular forms.

In this first section, we construct the relative modular curves and Igusa towers over the perfected weight space on which our perfectoid families of modular forms will live. In doing so, we will closely follow the constructions of [3], and our main task is to adapt these to a non-Noetherian setting, for which we rely on Noetherian approximation.

But before we can do so, the very first step is to introduce the perfected weight space. We will from now on usually work over a fixed weight space disc, since the finite part of the universal weight character is easy to deal with and will have no impact on our constructions.

6.1 Perfected weight space

Recall that the adic weight space for modular forms is $\mathcal{W}_{\text{full}} = \mathrm{Spa}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], \mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\text{an}}$, which is a disjoint union of $p-1$ open unit discs (or 2 discs if $p=2$) indexed by the finite characters $\mathbb{F}_p^\times \to \mathbb{Z}_p^\times$. As before, let us fix a weight space disc $\mathcal{W}$, then our chosen topological generator q of $1+p\mathbb{Z}_p$ (or $1+4\mathbb{Z}_2$ if $p=2$) defines an identification $\mathcal{W} = \mathrm{Spf}(\mathbb{Z}_p[[T]])_\eta^{\text{an}}$, for which the universal weight is given by sending $q \mapsto 1+T$. The group structure on the functor $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathcal{O}(-)^\times)$ gives $\mathcal{W}$ the structure of an adic group over $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, and we can canonically identify $\mathcal{W} = (\widehat{\mathbb{G}}_m)_\eta^{\text{an}}$ with the adic generic fibre of the completion at the identity of $\mathbb{G}_m \to \mathrm{Spf}(\mathbb{Z}_p)$. In terms of this group structure, the operation of sending a weight κ to $\kappa \mapsto \kappa^p$ is therefore described by the endomorphism of $\mathbb{Z}_p[[T]]$ sending $T \mapsto (T+1)^p - 1$.

We see from this explicit description that the space parametrising weights κ together with a choice of compatible p^n-th roots κ^{1/p^n} for all n is given by the generic fibre of

$$\mathfrak{W}^{\text{perf}} := \varprojlim_{\kappa \mapsto \kappa^p} \mathfrak{W} = \mathrm{Spf}((\varinjlim_{T \mapsto (1+T)^p - 1} \mathbb{Z}_p[[T]])^\wedge) = \mathrm{Spf}(\mathbb{Z}_p[[T]]\langle (1+T)^{1/p^\infty} \rangle).$$

where $C_\infty := \mathbb{Z}_p[[T]]\langle (1+T)^{1/p^\infty} \rangle$ is the (p,T)-adic completion of $\mathbb{Z}_p[[T]][(1+T)^{1/p^\infty}]$. More conceptually, in terms of the group structure, $\mathfrak{W}^{\text{perf}}$ is the universal cover $\varprojlim_{p\cdot} \mathfrak{W} = \varprojlim_{p\cdot} \widehat{\mathbb{G}}_m$ of $\mathfrak{W} = \widehat{\mathbb{G}}_m$ in the sense of Scholze–Weinstein [52]. Over $\mathfrak{W}^{\text{perf}}$ there is a universal p^n-th root weight κ^{1/p^n} defined by the function $\mathbb{Z}_p^\times \to C_\infty^\times$, $q \mapsto (1+T)^{1/p^n}$. We call this the "perfected weight space" since $\mathfrak{W}^{\text{perf}}$ is the inverse limit of canonical Frobenius lifts.

Since the reduction of $\mathbb{Z}_p[[T]]\langle (1+T)^{1/p^\infty} \rangle$ mod p can be canonically identified with $\mathbb{F}_p[[T]][1+T^{1/p^\infty}]^\wedge = \mathbb{F}_p[[T^{1/p^\infty}]]$, there is a canonical isomorphism

$$C_\infty := \mathbb{Z}_p[[T]]\langle (1+T)^{1/p^\infty} \rangle = W(\mathbb{F}_p[[T^{1/p^\infty}]]).$$

We note that there is also a canonical isomorphism $W(\mathbb{F}_p[[T^{1/p^\infty}]]) = \mathbb{Z}_p[[T^{1/p^\infty}]]$. This might look more friendly, but it does not have the correct $\mathbb{Z}_p[[T]]$-algebra structure for our purposes: We need elements of the form $(1+T)^{1/p^k}$ for the universal weights.

Notation 6.1.1. We would like to distinguish the role of T as the weight space parameter from the role that the image of T plays as a pseudo-uniformiser at the boundary, where we have $\mathbb{Z}_p[[T]]/p \cong \mathbb{F}_p[[T]]$. This is to avoid confusion when we later consider different weights valued in the same field K of characteristic p, and to be consistent with the perfectoid literature. We therefore make the following naming convention: We denote the image of T under reduction mod p by t, that is from now on we denote this reduction by $\mathbb{Z}_p[[T]] \to \mathbb{F}_p[[t]]$.

The following Lemma says that we may regard the universal weight over $\mathfrak{W}_\infty$ as being the canonical lift of the canonical weight $\overline{\kappa}$ at the boundary.

Lemma 6.1.2. *Composition with the lift* $[-] : \mathbb{F}_p[[t^{1/p^\infty}]] \to W(\mathbb{F}_p[[t^{1/p^\infty}]]) = C_\infty$ *sends the canonical boundary weight* $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t^{1/p^\infty}]]^\times$, $q \mapsto 1+t$, *to the universal weight*

$$\kappa : \mathbb{Z}_p^\times \to C_\infty^\times, \quad q \mapsto 1+T.$$

Proof. For any $k \in \mathbb{Z}_{\geq 0}$, the element $(1+T)^{1/p^k} \in C_\infty$ is a lift of $(1+t)^{1/p^k} \in \mathbb{F}_p[[t^{1/p^\infty}]]$. By definition of $[-]$, we thus have $[\overline{\kappa}(q)] = \lim_k ((1+T)^{1/p^k})^{p^k} = 1+T$ □

The ring C_∞ is clearly not Noetherian. Since some of the constructions in [3] require the base rings to be Noetherian, we will often need to approximate it by Noetherian algebras.

Definition 6.1.3. For any $k \in \mathbb{Z}_{\geq 0}$ we set $C_k := \mathbb{Z}_p[[T]][(1+T)^{1/p^k}]$. For any $k \leq k' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the map $\lambda_k^{k'} : C_k \to C_{k'}$ is clearly flat, and finite if $k' < \infty$. We also define

$$S_k := (1+T)^{1/p^k} - 1, \quad \text{then } S_k = (S_{k+1}+1)^p - 1. \tag{23}$$

Lemma 6.1.4. *For any $0 \leq j \leq k \in \mathbb{Z}_{\geq 0}$, we have $(p, S_j^{p^j}) \subseteq (p, T)$ and $(T) \subseteq (S_j)$ as ideals in C_k. In particular, the (p,T)-adic topology on C_k agrees with the (p, S_j)-adic one.*

Proof. In C_k/p, we have $S_j^{p^j} = ((1+T)^{1/p^j} - 1)^{p^j} = T$. This shows $(p, S_j^{p^j}) \subseteq (p, T)$. Conversely, we have $T = (S_j+1)^{p^j} - 1 = \sum_{i=1}^{p^j} \binom{p^j}{i} S_j^i \in (S_j)$, thus $T \in (S_j)$. $\square$

Definition 6.1.5. For any $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ we set

$$\mathcal{W}_k = \mathrm{Spa}(C_k, C_k)^{\mathrm{an}} = \mathrm{Spa}(\mathbb{Z}_p[[T]]\langle (1+T)^{1/p^k}\rangle, \mathbb{Z}_p[[T]]\langle (1+T)^{1/p^k}\rangle)^{\mathrm{an}}.$$

This is the "p^k-th root weight space": Its points on an analytic Huber pair (R, R^+) over $(\mathbb{Z}_p, \mathbb{Z}_p)$ correspond to weights $\kappa : \mathbb{Z}_p^\times \to R^\times$ with a choice of a p^k-th root κ^{1/p^k} of κ.

For any interval $I = [l, l'] \subseteq [0, \infty]$ with $l \leq l' \in 1/p^k\mathbb{Z}_{\geq 0}$, we define the open subspace

$$\mathcal{W}_{k,I} := \mathcal{W}_k(|p| \leq |S_k^{lp^k}| \neq 0, |S_k^{l'p^k}| \leq |p|).$$

When we picture $\mathcal{W}_k$ as a compactified disc, these are annuli whose inner radius is determined by l and whose outer radius is determined by l'.

For any $l \in \mathbb{Z}_{\geq 0}$, we also consider $I = [l, \infty]$ for which we define the open subspace

$$\mathcal{W}_{k,l} := \mathcal{W}_{k,I} := \mathcal{W}_k(|p| \leq |S_k^{lp^k}| \neq 0),$$
$$\mathcal{O}(\mathcal{W}_{k,l}) = \mathbb{Z}_p[[T]][(1+T)^{1/p^k}]\langle p/S_k^{lp^k}\rangle[1/S_k].$$

These are annuli in $\mathcal{W}_k$ containing the boundary whose inner radius increases with l.

We will almost exclusively work with the intervals $I = [l, \infty]$. But we need the more general definition for several technical steps in the comparison of modular sheaves with [3].

Definition 6.1.6. Let $k \in \mathbb{Z}_{\geq 0}$. The map sending $(\kappa, \kappa^{1/p^n}) \to \kappa^{1/p^n}$ defines an isomorphism of moduli problems $\mathcal{W}_k \xrightarrow{\sim} \mathcal{W}_0$. In terms of the group structure, this is multiplication by p^k. In terms of algebras, this corresponds to the "rescaling" isomorphism of $\mathbb{Z}_p$-algebras

$$r_k : C_0 = \mathbb{Z}_p[[T]] \xrightarrow{\sim} C_k, \quad T \mapsto S_k = (1+T)^{1/p^k} - 1, \quad (T+1)^{p^k} - 1 \mapsfrom T$$

which shows that $C_k = \mathbb{Z}_p[[S_k]]$. This is an isomorphism of topological algebras by Lemma 6.1.4.

Lemma 6.1.7. *For any $k \in \mathbb{Z}_{\geq 0}$ and $l \leq l' \in 1/p^k\mathbb{Z}_{\geq 0} \cup \{\infty\}$, $I = [l, l']$, set $p^k I := [lp^k, l'p^k]$. Then the rescaling isomorphism restricts to an isomorphism $r_k : \mathcal{W}_{k,I} \to \mathcal{W}_{0,p^kI}$. More generally, for any $d \leq k$ there is an isomorphism $r_d : \mathcal{W}_{k,I} \to \mathcal{W}_{k-d,p^dI}$, $S_k \mapsfrom S_{k-d}$.*

Proof. Rescaling by definition sends $S_k \mapsfrom T$ and thus for $l' < \infty$ identifies the subspaces

$$\mathcal{W}_k(|p| \leq |S_k^{lp^k}| \neq 0, |S_k^{l'p^k}| \leq |p| \neq 0) \xrightarrow{\sim} \mathcal{W}_0(|p| \leq |T^{lp^k}| \neq 0, |T^{l'p^k}| \leq |p| \neq 0).$$

Similarly for $I = [l, \infty]$. For the last statement, we set $r_d = r_{k-d}^{-1} \circ r_k$. $\square$

Remark 6.1.8. We need to be careful to dinstinguish T and S_k: While the topology on $\mathcal{W}_{k,l}$ is $(p, S_k) = (S_k)$-adic, it is also (p, T)-adic, but *not* in general (T)-adic. Furthermore, it is *not* true in general that T is a unit on $\mathcal{W}_{k,l}$, as one can see for $l = 1/p^k$ from the map

$$\mathbb{Z}_p[[T]][(1+T)^{1/p^k}]\langle p/S_k\rangle[1/S_k] \to \mathbb{Q}_p^{\mathrm{cyc}}, \quad S_k \mapsto \zeta_p - 1, \quad T \mapsto 0$$

Remark 6.1.9. For $k < \infty$, the space $\mathcal{W}_k$ is a Noetherian Tate analytic adic space, but it is not (the adification of) a rigid space in the classical sense since it does not live over a field. Instead, it can be covered by adic spectra of Tate algebras like $\mathbb{Z}_p[[T]]\langle p/T\rangle[1/T]$ that are "pseudo-affinoid algebras" in the sense of [43]. The space $\mathcal{W}_k$ is thus a "pseudo-rigid space" in the sense of [43] and [32], §2.5. In particular, it is sheafy.

Definition 6.1.10. We define formal models of $\mathcal{W}_{k,l}$ as follows: Let $C_{k,l} := C_k\langle p/S_k^{lp^k}\rangle = \mathbb{Z}_p[[T]]\langle(1+T)^{1/p^k}\rangle\langle u\rangle/(uS_k^{lp^k} - p)$ endowed with the (p,T)-adic topology. Then we set $\mathfrak{W}_{k,l} := \mathrm{Spf}(C_{k,l})$. More generally, for any interval of the form $I = [l, l']$ with $l \leq l' \in 1/p^k\mathbb{Z}_{\geq 0}$, we set $C_{k,I} := \mathbb{Z}_p[[T]]\langle(1+T)^{1/p^k}\rangle\langle u, v\rangle/(uS_k^{lp^k} - 1, vp - S_k^{l'p^k})$ and $\mathfrak{W}_{k,I} := \mathrm{Spf}(C_{k,I})$. It is clear from these definitions that we have $(\mathfrak{W}_{k,I})^{\mathrm{ad}}_\eta = \mathcal{W}_{k,I}$.

Definition 6.1.11. We also allow $l = l' = \infty$ for which we set $\mathfrak{W}_{k,\infty} := \mathrm{Spf}(C_{k,\infty})$ with $C_{k,\infty} = \mathbb{F}_p[[T^{1/p^k}]]$. Then $\mathcal{W}_{k,l} := \mathfrak{W}^{\mathrm{ad}}_{k,l}(T \neq 0) = \mathrm{Spa}(\mathbb{F}_p((T^{1/p^k})), \mathbb{F}_p[[T^{1/p^k}]])$ is simply a point. We have $S_k = T^{1/p^k}$ in this case. In the following, we will often allow $l \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to include this case. We see immediately from this Definition:

Lemma 6.1.12. *Let $k \in \mathbb{Z}_{\geq 0}$ and $l \leq l' \in 1/p^k\mathbb{Z}_{>0} \cup \{\infty\}$ set $I = [l, l']$. The rescaling map has a formal model $r_d : \mathfrak{W}_{k,I} \xrightarrow{\sim} \mathfrak{W}_{k-d,p^dI}$ given by $S_k \mapsfrom S_{k-d}$ that is also an isomorphism.*

Lemma 6.1.13. *For any $k \in \mathbb{Z}_{\geq 0}$ and any $l \leq l' \in 1/p^k\mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $I = [l, l']$, the space $\mathfrak{W}_{k,I}$ is normal. In particular, for any open formal subscheme $U \subseteq \mathfrak{W}_{k,I}$, we have*

$$\mathcal{O}^+_{\mathcal{W}_k}(U) = \mathcal{O}_{\mathfrak{W}_{k,I}}(U).$$

Proof. To prove that $\mathfrak{W}_{k,I}$ is normal it suffices to prove that $C_{k,I}$ is normal. After rescaling, it suffices to prove that $C_{0,I}$ is normal for $l, l' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, which one can verify using Serre's criterion as discussed in [3], §3.3.1. The last statement then follows by Lemma A.2.5. $\square$

The following Lemma is not hard to prove, but we postpone the proof to Lemma 8.3.8.

Lemma 6.1.14. *The space $\mathcal{W}_{\infty,I} := (\mathfrak{W}_{\infty,I})^{\mathrm{ad}}_\eta$ is a sheafy adic space.*

Although the definition of the radii in the annulus $\mathcal{W}_{k,I}$ is made in terms of the variable S_k, the following Lemma says that the radius only depends on l and is independent of k. Namely, the Lemma says that "away from the centre of $\mathcal{W}$", we have $|S_k| = |S_{k+1}|^p$.

Lemma 6.1.15. *Let $m, k \in \mathbb{Z}_{\geq 0}$ and $1 \leq j \leq k$. Then*

1. *We have $\mathcal{W}_k(|p| \leq |S^m_{j-1}| \neq 0) = \mathcal{W}_k(|p| \leq |S^{mp}_j| \neq 0)$ as rational subspaces of $\mathcal{W}_k$.*
2. *If we let $I = [l, l']$ for any $l \leq l' \in 1/p^k\mathbb{Z}_{>0} \cup \{\infty\}$, then for any $k' \geq k$ there is a natural map $\mathcal{W}_{k',I} \to \mathcal{W}_{k,I}$ and we have $(S_k) = (S_{k'}^{p^{k'-k}})$ in $C_{k',I}$.*

Proof. Since S_j is topologically nilpotent, we have $|S_j| < 1$. By equation (23), one can write

$$S_{j-1} = (S_j + 1)^p - 1 = S_j^p + pS_j^{p-1} + \cdots + pS_j.$$

Let us set $\delta := S_{j-1} - S_j^p = pS_j^{p-1} + \cdots + pS_j$, then $|\delta| \leq |pS_j| < |p|$ since $|S_j| < 1$.

If now $|p| \leq |S_{j-1}|$, then $|\delta| < |p| \leq |S_{j-1}|$ which implies $|S_j^p| = |S_{j-1} - \delta| = |S_{j-1}|$. Conversely, if $|p| \leq |S_j^p|$, then $|\delta| < |p| \leq |S_j^p|$ and thus $|S_{j-1}| = |S_j^p + \delta| = |S_j^p|$. This shows that $\mathcal{W}_k(|p| \leq |S_{j-1}| \neq 0) = \mathcal{W}_k(|p| \leq |S_j^p| \neq 0)$. Since $\mathcal{W}_k(|p| \leq |S^m_{j-1}| \neq 0)$ is a subspace of this, we may simply raise $|S_{j-1}| = |S_j^p|$ to the m-th power to deduce the first part.

For part 2, it suffices by induction to consider the case of $k' = k+1$. The existence of $\mathcal{W}_{k+1,I} \to \mathcal{W}_{k,I}$ is then a consequence of the first part where we plug in $k+1$, k and $p^k l$ for what is there denoted by k, j and m. In particular, the bounded function p/S_k on the $\mathcal{W}_{k,I}$ defines an element $p/S_k \in C_{k+1,l}$. We may therefore rewrite the above equation as

$$S^p_{k+1} = S_k(1 - pS_{k+1}/S_k \cdot (\dots)).$$

Since p is topologically nilpotent, so is pS_{k+1}/S_k, and thus the factor $(1 + pS_j/S_{j-1} \cdot (\dots))$ is a unit in $\mathcal{O}(\mathfrak{W}_{k,l})$. This gives the desired equality $(S_k) = (S^p_{k+1})$. $\square$

Notation 6.1.16. Let $m \in \mathbb{Z}_{\geq 0}$, $l \in p^{-m}\mathbb{Z}_{>0}$.

1. Let $\delta \in \mathbb{Z}[1/p]_{\geq 0}$. Then by Lemma 6.1.15.2, for any $m \leq k \leq k' \in \mathbb{Z}_{\geq 0}$ with $p^k\delta \in \mathbb{Z}_{\geq 0}$, the ideal $(S_k^{\delta p^k})$ in $C_{k',l}$ is independent of k. We will denote this ideal by $(S^\delta) := (S_k^{\delta p^k})$.

2. For $k = \infty$, we define for any $\delta \in \mathbb{R}$ a $C_{\infty,l}$-submodule of $C_{\infty,l}[1/S_m]$ by setting

$$(S^\delta) := \bigcup_{d \geq \delta, d \in \mathbb{Z}[1/p]} (S^d) \subseteq C_{\infty,l}[1/S_m].$$

   To keep notation uniform, set $S_\infty^\delta := S^\delta$. Note that $(S^{\delta_1}) \subseteq (S^{\delta_2})$ for any $\delta_1 \geq \delta_2 \in \mathbb{R}$.

3. Similarly, for any $k \geq m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and for any $C_{k,l}$-module M we have $M[1/S_m] = M[1/S_{m'}]$ for any $m \leq m'$ and we denote this module by $M[1/S]$. In particular, it makes sense to speak of the S-adic generic fibre of a formal scheme over $\mathfrak{W}_{k,l}$. We note that this does *not* work for $m' < m$, and we typically have $M[1/S] \neq M[1/T]$.

We will also need the following estimates complementing Lemma 6.1.15

Lemma 6.1.17. *Let $k \in \mathbb{Z}_{\geq 0}$ and $x \in \mathcal{W}_k$, then for any $0 \leq m < k$,*

1. $|S_{m+1}(x)| > |p|^{1/(p-1)}$ *implies* $|S_m(x)| = |S_{m+1}(x)^p|$,
2. $|S_{m+1}(x)| = |p|^{1/(p-1)}$ *implies* $|S_m(x)| \leq |S_{m+1}(x)^p| < |p| \leq |p|^{1/(p-1)}$,
3. $|S_{m+1}(x)| < |p|^{1/(p-1)}$ *implies* $|S_m(x)| < |p| \leq |p|^{1/(p-1)}$.

In particular, we have $\mathcal{W}_k(|p| \geq |S_k^{p^k}|) \subseteq \mathcal{W}_k(|p| \geq |T|)$.

Proof. Recall that for any $0 \leq m < k$, we have

$$S_m = S_{m+1}^p + pS_{m+1}^{p-1} + \cdots + \tbinom{p}{2}S_{m+1}^2 + pS_{m+1} = S_{m+1}^p + pS_{m+1} \cdot x$$

where $x = 1 + S_{m+1}(\dots) \in C_k^\times$ is a unit since S_{m+1} is topologically nilpotent. Consequently,

1. In case 1 we have $|S_{m+1}^p(x)| > |pS_{m_1}(x)|$. Thus $|S_m(x)| = |S_{m+1}^p(x)|$.
2. In case 2, $|S_m(x)| \leq \max(|S_{m+1}^p(x)|, |pS_{m+1}^p(x)|) \leq |S_{m+1}^p(x)|$.
3. In case 3, $|S_{m+1}^p(x)| < |pS_{m_1}(x)|$. Then $|S_m(x)| = |S_{m+1}^p(x)| < |p|^{p/(p-1)} < |p|$.

This shows that either we are in case 3 for some $0 \leq m < k$, in which case we are also in case 3 for $m-1, \dots, 0$ and thus $|T(x)| = |S_0| < |p|$. Or we are in cases 1,2 for all $0 \leq m < k$, in which case $|S_k^{p^k}(x)| \leq |p|$ implies $|T(x)| = |S_0(x)| \leq |S_1^p(x)| \leq |S_2^{p^2}(x)| \leq \cdots \leq |S_k^{p^k}(x)| \leq |p|$. □

Definition 6.1.18. For any $I_1 = [l_1, l_1'] \subseteq I_0 = [l_0, l_0'] \subseteq [0, \infty]$, there is a natural morphism $\lambda_{I_0}^{I_1} : \mathcal{W}_{k,I_1} \to \mathcal{W}_{k,I_0}$: If $l_1 < \infty$, this is a restriction to an open rational subspace. For $l_1 = \infty$, i.e. for $I_1 = [\infty, \infty] \subseteq [l, \infty]$, it is defined by the reduction map $C_{k,l} \to \mathbb{F}_p[[t^{1/p^k}]]$, $S_k \mapsto t^{1/p^k}$. By restriction to the $\mathcal{O}^+$-sheaves one obtains a formal model $\mathfrak{W}_{k,I_1} \to \mathfrak{W}_{k,I_0}$.

Lemma 6.1.19. *For $k \leq k' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $l \in p^{-k}\mathbb{Z}_{>0} \cup \{\infty\}$, the map $\lambda_k^{k'} : \mathcal{W}_{k',l} \to \mathcal{W}_{k,l}$ from Lemma 6.1.15.2 has a formal model $\lambda_k^{k'} : \mathfrak{W}_{k',l} \to \mathfrak{W}_{k,l}$. It fits into a Cartesian diagram*

$$\begin{array}{ccc} \mathfrak{W}_{k'} & \longrightarrow & \mathfrak{W}_k \\ \uparrow & & \uparrow \\ \mathfrak{W}_{k',l} & \longrightarrow & \mathfrak{W}_{k,l}. \end{array}$$

Proof. To construct the diagram, the morphism $\lambda_k^{k'} : \mathfrak{W}_{k',l} \to \mathfrak{W}_{k,l}$ is induced from taking sections of $\lambda_k^{k'} : \mathcal{W}_{k',l} \to \mathcal{W}_{k,l}$ on the $\mathcal{O}^+$-sheaf by Lemma 6.1.13. To see that the diagram on the right is Cartesian, we need to see that the natural morphism

$$C_k\langle p/S_k^{lp^k}\rangle \otimes_{C_k} C_{k+1} = C_{k+1}\langle p/S_k^{lp^k}\rangle \to C_{k+1,l} = C_{k+1}\langle p/S_{k+1}^{lp^{k+1}}\rangle$$

is an isomorphism, which follows from Lemma 6.1.15.2.

In the limit $k' \to \infty$ over the diagram, we get the desired statement for $k' = \infty$. □

The following is a prototypical example of what we mean by "Noetherian approximation":

Lemma 6.1.20. *Let $l \leq l' \in \mathbb{Z}_{\geq 0}[1/p] \cup \{\infty\}$ with $I = [l, l']$. Then $\mathfrak{W}_{\infty,I}$ is integrally closed in its generic fibre, that is the sheaf of subrings $\mathcal{O}_{\mathfrak{W}_{\infty,I}} \subseteq \mathcal{O}_{\mathfrak{W}_{\infty,I}}[1/S]$ is integrally closed.*

Proof. By Lemma 6.1.13, $\mathfrak{W}_{k,I}$ is integrally closed in its S_k-adic generic fibre for $k < \infty$. Let $k \geq 0$ be such that $p^k l \in \mathbb{Z}$. Then by Lemma 6.1.15, for $k' \geq k$, the $S_{k'}$-adic and S_k-adic generic fibres coincide. In the limit $k \to \infty$, the Lemma now follows from Lemma A.2.2.1. □

Remark 6.1.21. One can prove that for any $l \in p^{-k}\mathbb{Z}_{>0} \cup \{\infty\}$, the ring $C_{k,l}[1/S]$ is a principal ideal domain. This can be used to see that the extended eigencurve is flat over $\mathcal{W}$.

6.2 Modular curves of tame level

We now follow [3] §3 in introducing various rigid modular curves and canonical formal models. To be consistent with the earlier sections, our notation will be slightly different. Our goal is ultimately to work over $\mathcal{W}_{\infty,l}$, but like before we need to work by Noetherian approximation. We also discuss transition maps in the weight space parameter l and Frobenius lifts, as well as Igusa curves. Nothing in this section is in any way conceptually new or difficult. Instead, most of it amounts to bookkeeping that we need to carry out carefully in order to be able to work freely with objects over the perfected weight space in later sections.

Definition 6.2.1. Let $k \in \mathbb{Z}_{\geq 0}$, $l \leq l' \in p^{-k}\mathbb{Z}_{>0} \cup \{\infty\}$, $r \in \mathbb{Z}$. Set $I = [l, l'] \subseteq [0, \infty]$.

1. Let $\mathfrak{X}_k := \mathfrak{X}_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} \mathfrak{W}_k \to \mathfrak{W}_k$ be the (p, T)-adically completed compactified modular curve of tame level N over $C_k = \mathbb{Z}_p[[T]][(1+T)^{1/p^k}]$.
2. Let $\mathfrak{X}_{k,I} := \mathfrak{X}_k \times_{\mathfrak{W}_k} \mathfrak{W}_{k,I} = \mathfrak{X}_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} \mathfrak{W}_{k,I}$. This is the (p, T)-adically (or equivalently (S_k)-adically) completed compactified modular curve of tame level N over $C_{k,I} = \mathbb{Z}_p[[T]]\langle(1+T)^{1/p^k}\rangle\langle p/S_k^{lp^k}, S_k^{l'p^k}/p\rangle$. Like before, we set $\mathfrak{X}_{k,l} := \mathfrak{X}_{k,[l,\infty]}$.
3. Let $\mathcal{X}_{k,I} := \mathfrak{X}_{k,I} \times_{\mathfrak{W}_{k,I}} \mathcal{W}_{k,I}$ be the S_k-adic generic fibre. Again $\mathcal{X}_{k,l} := \mathcal{X}_{k,[l,\infty]}$.
4. For $r \geq 0$, let $\mathcal{X}_{r,k,I} := \mathcal{X}_{k,I}(|S_k^{p^k}| \leq |\mathrm{Ha}^{p^{r+1}}| \neq 0) \subseteq \mathcal{X}_{k,I}$, where as usual to make this precise we choose local lifts of the Hasse invariant. We let $\mathcal{X}_{r,k,l} := \mathcal{X}_{r,k,[l,\infty]}$.
5. The adic space $\mathcal{X}_{r,k,I}$ has a canonical S-adic formal model defined as follows: For any open $U = \operatorname{Spf} R \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ over which ω is trivial, let Ha denote the Hasse invariant if $l = l' = \infty$, and else a lift of the Hasse invariant. Then $\mathfrak{X}_{r,k,I|U} := \operatorname{Spf}(A_{r,k,I,U})$ where

$$A_{r,k,I,U} := \begin{cases} R\hat{\otimes}_{\mathbb{Z}_p} C_{k,I}\langle X\rangle/(X\mathrm{Ha}^{p^{r+1-k}} - S_k) & \text{if } r+1 \geq k \\ R\hat{\otimes}_{\mathbb{Z}_p} C_{k,I}\langle X\rangle/(X\mathrm{Ha} - S_k^{p^{k-r-1}}) & \text{if } r+1 \leq k. \end{cases}$$

   This definition glues to give the desired formal scheme $\mathfrak{X}_{r,k,I}$. We set $\mathfrak{X}_{r,k,l} := \mathfrak{X}_{r,k,[l,\infty]}$ and $A_{r,k,l,U} := A_{r,k,[l,\infty],U}$. We will often omit U from notation.

Lemma 6.2.2. *For any $d \leq k$, rescaling $r_k : \mathcal{W}_{k,I} \xrightarrow{\sim} \mathcal{W}_{k-d,p^dI}$ induces an $\mathfrak{X}_{\mathbb{Z}_p}$-linear isomorphism $r_k : \mathcal{X}_{k,I} \xrightarrow{\sim} \mathcal{X}_{k-d,p^dI}$ which restricts to an isomorphism of subspaces $r_d : \mathcal{X}_{r,k,I} \xrightarrow{\sim} \mathcal{X}_{r-d,k-d,p^dI}$. This has a canonical formal model $r_d : \mathfrak{X}_{r,k,I} \xrightarrow{\sim} \mathfrak{X}_{r-d,k-d,p^dI}$.*

Proof. Since r_d is $\mathbb{Z}_p$-linear, we may base change the rescaling map $r_d : \mathfrak{W}_{k,I} \xrightarrow{\sim} \mathfrak{W}_{k-d,p^dI}$ of weight space along $\mathfrak{X}_{\mathbb{Z}_p} \to \mathrm{Spf}(\mathbb{Z}_p)$ to obtain an isomorphism $r_d : \mathfrak{X}_{k,I} \xrightarrow{\sim} \mathfrak{X}_{k-d,p^dI}$. The generic fibre of this gives the desired isomorphism $r_d : \mathcal{X}_{k,I} \xrightarrow{\sim} \mathcal{X}_{k-d,p^dI}$. To see that this restricts to the desired subspaces, we note that r_d by definition fixes Ha and thus restrict to

$$r_d : \mathcal{X}_{k,I}(|S_k^{p^k}| \leq |\mathrm{Ha}^{p^{r+1}}| \neq 0) \xrightarrow{\sim} \mathcal{X}_{k-d,p^dI}(|S_{k-d}^{p^k}| \leq |\mathrm{Ha}^{p^{r+1}}| \neq 0) = \mathcal{X}_{r-d,k-d,p^dI}$$

as $|S_{k-d}^{p^k}| \leq |\mathrm{Ha}^{p^{r+1}}|$ if and only if $|S_{k-d}^{p^{k-d}}| \leq |\mathrm{Ha}^{p^{r-d+1}}|$. For the formal models, we compare the definitions in terms of the $A_{r,k,I}$: For $r+1 < k$, the map $r_d : \mathfrak{X}_{k,I} \xrightarrow{\sim} \mathfrak{X}_{k-d,p^dI}$ induces

$$r_d : A_{r-d,k-d,p^dI} \to A_{r,k,I}, \quad X\mathrm{Ha}^{p^{(r-d)+1-(k-d)}} - S_{k-d} \mapsto X\mathrm{Ha}^{p^{r-k+1}} - S_k.$$

For $r+1 \geq k$, it similarly sends $r_d(X\mathrm{Ha} - S_{k-d}^{p^{(k-d)-(r-d)-1}}) = X\mathrm{Ha} - S_k^{p^{k-r-1}}$. □

We record the following algebraic properties of these modular curves, which are all either immediate or follow from the respective statements in [3] by rescaling:

Notation 6.2.3. In the following, we are going to discuss transition maps of the form $\mathfrak{X}_{r',k',I'} \to \mathfrak{X}_{r,k,I}$. Since we only ever consider one kind of transition map for each index, and these transition maps are commutative in the different indexes, we will not give these maps names, since it will be clear from the indices what the respective transition is.

Lemma 6.2.4. *1. The formal scheme $\mathfrak{X}_{r,k,I}$ is normal and excellent.*

2. For any $k' \geq k$, the base change $\mathcal{W}_{k',I} \to \mathcal{W}_{k,I}$ induces a natural map $\mathcal{X}_{k',I} \to \mathcal{X}_{k,I}$ which restricts to the subspaces $\mathcal{X}_{r,k',I} \to \mathcal{X}_{r,k,I}$. These maps have canonical formal models $\mathfrak{X}_{k',I} \to \mathfrak{X}_{k,I}$ and $\mathfrak{X}_{r,k',I} \to \mathfrak{X}_{r,k,I}$.

3. For any $k' \geq k \geq r+1$ and any $I' \subseteq I$, there are Cartesian diagrams

$$\text{(i)}\ \begin{array}{ccc} \mathfrak{X}_{r,k',I} & \longrightarrow & \mathfrak{X}_{r,k,I} \\ \downarrow & & \downarrow \\ \mathfrak{W}_{k',I} & \longrightarrow & \mathfrak{W}_{k,I}, \end{array} \quad \text{(ii)}\ \begin{array}{ccc} \mathfrak{X}_{r,k,I'} & \longrightarrow & \mathfrak{X}_{r,k,I} \\ \downarrow & & \downarrow \\ \mathfrak{W}_{k,I'} & \longrightarrow & \mathfrak{W}_{k,I}, \end{array} \quad \text{(iii)}\ \begin{array}{ccc} \mathfrak{X}_{r,k',I'} & \longrightarrow & \mathfrak{X}_{r,k,I'} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{r,k',I} & \longrightarrow & \mathfrak{X}_{r,k,I}. \end{array}$$

Proof. Excellence in 1 follows from the main result of [57]. Normality for $r+1 \geq k$ follows by rescaling from [3], Lemme 3.4. For $r+1 \leq k$, it is implicit in the discussion in [3] §6.3.2-6.3.3. We are going to prove a slightly more general statement in Lemma 8.3.5 below.

The map $\mathcal{X}_{k',I} \to \mathcal{X}_{k,I}$ is the base change of $\mathcal{W}_{k',I} \to \mathcal{W}_{k,I}$ along $\mathcal{X}_{k,I} \to \mathcal{W}_{k,I}$. The statements then follow from Lemma 6.1.15.2. The existence of formal models follows from the explicit definitions.

Part 3(i) and (ii) are also immediate from the explicit definitions. Part (iii) follows from (i) from a Cartesian cube, using that $\mathfrak{W}_{k',I'} = \mathfrak{W}_{k',I} \times_{\mathfrak{W}_{k,I}} \mathfrak{W}_{k,I'}$. □

6.3 Igusa curves

In this section we discuss canonical subgroups and Igusa curves over the modular curves of the last section. We are still following [3], §3 which in our setting treats the case of $k = 0$.

In this subsection, we let $k, r \in \mathbb{Z}_{\geq 0}$. We will only pass to $k \to \infty$ later. From now on we will work exclusively over regions of weight space of the form $\mathcal{W}_{k,l} := \mathcal{W}_{k,I}$ with $I = [l, \infty]$ for some $l \in p^{-k}\mathbb{Z}_{>0} \cup \{\infty\}$, that is we consider annuli containing the boundary.

In the following, we want to bound the inner radius variable l independently of k. We therefore introduce a variable $m \le k$ whose purpose is that we assume $l \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$.

On $\mathfrak{X}_{r,k,l}$, there is a universal semi-abelian scheme $\mathfrak{E}_{r,k,l} \to \mathfrak{X}_{r,k,l}$ obtained by pullback along $\mathfrak{X}_{r,k,l} \to \mathfrak{X}_{k,l} \to \mathfrak{X}_{\mathbb{Z}_p}$. Since the rescaling isomorphism $r_k : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r-k,0,lp^k}$ is $\mathbb{Z}_p$-linear, it induces an isomorphism $r_k : \mathfrak{E}_{r,k,l} \xrightarrow{\sim} \mathfrak{E}_{r-k,0,lp^k}$ lying over $r_k : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r-k,0,lp^k}$.

We recall from [3], §A the definition of the Hodge ideal:

Definition 6.3.1. Let us simply denote by ω the sheaf of invariant differentials of the semi-abelian scheme on $\mathfrak{X}_{r,k,l}$. Recall that the Hasse invariant Ha is a section of $\omega^{\otimes(p-1)}$ on the reduction of $\mathfrak{X}_{r,k,l}$ mod p. We denote by $\mathrm{Hdg}_{r,k,l}$ the subsheaf of ideals of $\mathcal{O}_{\mathfrak{X}_{r,k,l}}$ defined as the preimage under the reduction of the sheaf of ideals $\mathrm{Ha} \cdot \omega^{\otimes(1-p)} \subseteq \mathcal{O}_{\mathfrak{X}_{r,k,l}}/p$.

Assume now that $r \ge m+1$ and let $n \le r-m$. With this setup, we have $p \in \mathrm{Hdg}_{r,k,l}^{p^{n+1}}$ and thus [3], Corollaire A.2 shows that the semi-abelian scheme has a canonical subgroup $H_{n,r,k,l}$ of rank p^n, which is a finite flat subgroup scheme of $\mathfrak{E}_{r,k,l}$. Its dual $H^\vee_{n,r,k,l}$ becomes finite étale after inverting S_k, and is étale locally isomorphic to $\underline{\mathbb{Z}/p^n\mathbb{Z}}$.

Alternatively, $H_{n,r,k,l}$ may be constructed by rescaling from the case of $k=0$ discussed in [3]: For this we simply define it as the pullback of $H_{n,r-m,0,lp^m}$ along the morphism

$$\mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r,m,l} \xrightarrow{r_m} \mathfrak{X}_{r-m,0,lp^m}. \tag{24}$$

Definition 6.3.2. For any $k \in \mathbb{Z}_{\ge 0}$, $l \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$, $r \ge m+1$ and $0 \le n \le r-m$, let

$$\mathcal{IG}_{n,r,k,l} \to \mathcal{X}_{r,k,l}$$

be the finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor which relatively represents the choice of an isomorphism $\underline{\mathbb{Z}/p^n\mathbb{Z}} \xrightarrow{\sim} H^\vee_{n,r,k,l}$. For simplicity of notation, we set $\mathcal{IG}_{0,r,k,l} := \mathcal{X}_{r,k,l}$.

It is clear from the moduli problem that for any $n' \ge n$ the projection $\mathbb{Z}/p^{n'}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ induces a natural forgetful morphism $\mathcal{IG}_{n',r,k,l} \to \mathcal{IG}_{n,r,k,l}$ over $\mathcal{X}_{r,k,l}$.

Definition 6.3.3. Let $\mathfrak{IG}_{n,r,k,l} \to \mathfrak{X}_{r,k,l}$ be the normalisation of $\mathfrak{X}_{r,k,l}$ in $\mathcal{IG}_{n,r,k,l} \to \mathcal{X}_{r,k,l}$. One sees exactly like in [3] Lemme 3.2 that this is well-defined, and gives a finite morphism. Again, we include the case of $n=0$ and set $\mathfrak{IG}_{0,r,k,l} := \mathfrak{X}_{r,k,l}$.

Notation 6.3.4. Extending the notation from Definition 6.2.1.5, we shall for any $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ set $B_{n,r,k,l,U} := \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}(U)$. In particular, we have a finite ring extension $A_{r,k,l,U} \to B_{n,r,k,l,U}$ that is integrally closed in its generic fibre. We often drop U from the notation.

We wish to apologise to the reader for the quadruple index, but we really need flexibility in all four parameters. We hope to make good for that with Figure 1, which is supposed to make it easier for the reader to go back to see what each variable stands for.

Lemma 6.3.5. *$\mathfrak{IG}_{n,r,k,l}$ is a normal excellent formal scheme. In particular, for the natural map of ringed spaces $s : \mathcal{IG}_{n,r,k,l} \to \mathfrak{IG}_{n,r,k,l}$, we have $s_*\mathcal{O}^+_{\mathcal{IG}_{n,r,k,l}} = \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}$.*

Proof. The first part is clear from the definition, the second follows from Lemma A.2.4. □

By the universal property of the normalisation, the map $\mathcal{IG}_{n',r,k,l} \to \mathcal{IG}_{n,r,k,l}$ has a formal model $\mathfrak{IG}_{n',r,k,l} \to \mathfrak{IG}_{n,r,k,l}$, which is finite since $\mathfrak{IG}_{n,r,k,l}$ is excellent. Similarly:

Lemma 6.3.6. *For any $k \le k' \in \mathbb{Z}_{\ge 0}$, $l \le l' \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$, $r \ge m+1$ and $n \le r-m$, $\mathcal{IG}_{n,r,k',l'} \to \mathcal{X}_{r,k',l'}$ is the base change of $\mathcal{IG}_{n,r,k,l} \to \mathcal{X}_{r,k,l}$ along $\mathcal{X}_{r,k',l'} \to \mathcal{X}_{r,k,l}$. There is a canonical formal model $\mathfrak{IG}_{n,r,k',l'} \to \mathfrak{IG}_{n,r,k,l}$ which commutes with the transition maps in n. In the case that $l'=l$, the map $\mathfrak{IG}_{n,r,k',l} \to \mathfrak{IG}_{n,r,k,l}$ is a finite morphism.*

Remark 6.3.7. For $l=\infty$, one can describe this normalisation explicitly in terms of the Igusa curve $X_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \to X_{\mathbb{F}_p}$: By [3], Lemme 4.1, it is $\mathfrak{IG}_{n,r,k,\infty} = X_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \times_{X_{\mathbb{F}_p}} \mathfrak{X}_{r,k,\infty}$.

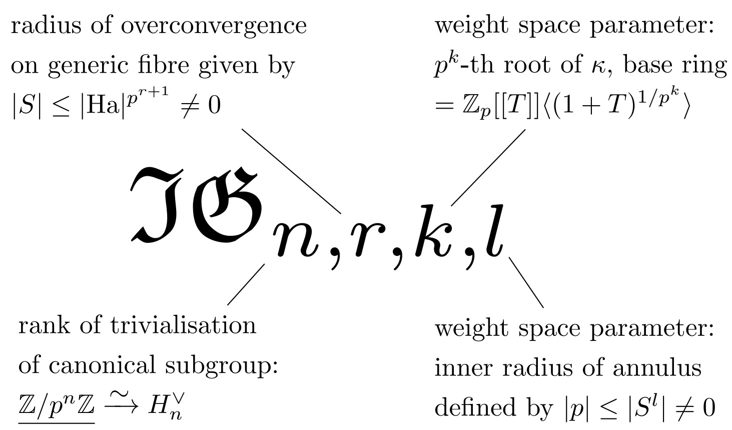

Figure 1: overview of the meaning of the indices in $\mathfrak{IG}_{n,r,k,l}$

6.4 Frobenius lifts

The operation of dividing by the canonical subgroup defines a morphism of adic spaces $\mathcal{X}_{r,k,l} \to \mathcal{X}_{r-1,k,l}$, as we shall now discuss. This is our transition map in the variable r.

Proposition 6.4.1. *Let $m \leq k \leq k' \in \mathbb{Z}_{\geq 0}$, $l \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$ and $r \geq m+1$.*

1. *The isogeny defined by the canonical subgroup $H_{1,r,k,l} \hookrightarrow \mathfrak{E}_{r,k,l}$ induces a finite and flat morphism $\phi : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r-1,k,l}$ which fits into a Cartesian diagram*

$$\begin{tikzcd} \mathfrak{E}_{r,k,l}/H_{1,r,k,l} \arrow[r] \arrow[d] & \mathfrak{E}_{r-1,k,l} \arrow[d] \\ \mathfrak{X}_{r,k,l} \arrow[r, "\phi"] & \mathfrak{X}_{r-1,k,l}. \end{tikzcd}$$

2. *If $k \geq r+1$, then $\phi : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r-1,k,l}$ reduces mod $p/S_k^{(p+1)p^{k-r-1}}$ to the Frobenius relative to $\mathfrak{W}_{k,l}$. If $k < r+1$, this is only true after inverting S.*

3. *The map ϕ_r commutes with the transition maps in l and k. Moreover, for $k \geq r+1$, $\mathfrak{X}_{r+1,k',l} \to \mathfrak{X}_{r,k',l}$ is the base-change of $\phi : \mathfrak{X}_{r+1,k,l} \to \mathfrak{X}_{r,k,l}$ along $\mathfrak{X}_{r,k',l} \to \mathfrak{X}_{r,k,l}$.*

Proof. For the proof, we follow [3], Proposition 3.3: The case of $k < r$ follows by applying the Proposition with $A_0 = C_{k,l}$ and $\alpha = S_k$ and "r"$= r-k$. The formal scheme denoted there by "$\mathfrak{Y}_r$" is then precisely our $\mathfrak{X}_{r,k,l}$, and we obtain part 1 of the Proposition. This gives the generic fibre of ϕ a moduli description, which immediately implies parts 2 and 3.

The case of $k \geq r$ for parts 1 and 3 follows from this by base-change: Since $r \geq m+1$, we may assume after rescaling via r_m that $l \in \mathbb{Z}_{\geq 0}$. The desired statements then follow on the generic fibre from the case of $k = 0$ by base-change along $\mathcal{W}_{k,l} \to \mathcal{W}_{0,l}$. The statements for formal models follow form this by normalisation, using that ϕ is finite and $\mathfrak{X}_{r,k,l}$ is normal.

It remains to prove part 2 in the case of $k \geq r+1$: One way to do this is by slightly modifying the proof of [3], Proposition 3.3.

Like in [50], Definition III.2.12, we can regard $\mathfrak{X}_{r,k,l}$ as the formal scheme over $C_{k,l}$ representing the functor which sends a $C_{k,l}$-algebra R to the set of isomorphism classes of pairs (f, η), where f is a morphism $f : \mathrm{Spf}(R) \to \mathfrak{X}_{k,l}$ giving rise to a semi-abelian scheme E over R, and $\eta \in \omega_E^{(1-p)}$ is such that $\mathrm{Ha}(E)\eta \equiv S_k^{p^{k-r-1}}$ mod p, and where (f, η) and (f', η') are equivalent if $f = f'$ and if there is $h \in R$ with $\eta = \eta'(1 + \pi \cdot h)$ where $\pi := p/S_k^{p^{k-r-1}}$.

Since the Hodge ideal Hdg of $\mathfrak{X}_{r,k,l}$ contains $S_k^{p^{k-r-1}}$, and $p = (p/S_k^{lp^k}) \cdot S_k^{lp^k} \in (S_k^{p^{k-r}})$ by $l \geq p^{-m} \geq p^{-r}$, there is on $\mathfrak{X}_{r,k,l}$ a canonical subgroup H_1 by [3], Corollaire A.2.1, which reduces to the kernel of Frobenius modulo p/Hdg. Since $S_k^{p^{k-r-1}} \in \mathrm{Hdg}$, we have $p/\mathrm{Hdg} \subseteq (p/S_k^{p^{k-r-1}}) = (\pi)$, and thus H_1 also reduces to the kernel of Frobenius mod π.

We can now define ϕ as follows: Set $E' = E/H_1(E)$, then E reduces mod π to $E^{(p)}$. We therefore have $\omega_{E'} \equiv \omega_E^p \bmod \pi$ and $\mathrm{Ha}(E') \equiv \mathrm{Ha}(E)^p \bmod \pi$. But then since $p \in (\pi)$,

$$\mathrm{Ha}(E)\eta \equiv S_k^{p^{k-r-1}} \bmod p \quad \Rightarrow \quad \mathrm{Ha}(E')\eta^p \equiv \mathrm{Ha}(E)^p\eta^p \equiv S_k^{p^{k-r}} \bmod \pi.$$

Let us assume without loss of generality that ω_E is free over R. We choose a trivialisation so that we may write $\mathrm{Ha}(E') \in R$. We claim that one can then find $\eta' \in \omega_{E'}^{(1-p)}$ such that

$$\eta' \equiv \eta^p \bmod \pi' \quad \text{ and } \quad \mathrm{Ha}(E') \cdot \eta' = S_k^{p^{k-r}} \bmod p \quad \text{ where } \quad \pi' := p/S_k^{(p+1)p^{k-r-1}}.$$

Indeed let η'' be any lift of η^p to $\omega_{E'}$, then as $\eta'' \equiv \eta^p \bmod \pi$, there is $x \in R$ such that

$$\mathrm{Ha}(E')\eta'' = S_k^{p^{k-r}} + \pi x = S_k^{p^{k-r}}(1 + p/S_k^{p^{k-r-1}+p^{k-r}}x) = S_k^{p^{k-r}}(1+\pi' x).$$

Since $r \geq m+1$, we have $(p+1)p^{k-r-1} < p^{k-r+1} < p^m$, thus $\pi' \in S_k \cdot \frac{p}{S_k^{p^{k-m}}}$, which implies that $(1+\pi' x)$ is a unit in R. Let us now set

$$\eta' := (1+\pi' x)^{-1}\eta'' \in \omega_{E'},$$

then since $\pi \in (\pi')$, we have $\eta' \equiv \eta'' \equiv \eta^p \bmod \pi'$ as desired.

Let now $f' : \mathrm{Spf}(R) \to \mathfrak{X}_{k,l}$ be the map corresponding to E', then we define $\phi(f, \eta) := (f', \eta')$. Since the lift x above was uniquely determined up to a factor of the form $(1 + \pi h)$ for some $h \in R$, the same is true for η'. This shows that ϕ is functorial in R and thus defines

$$\phi : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r-1,k,l}.$$

To see that ϕ reduces to the relative Frobenius mod π', we note that ϕ is locally of the form

$$\phi : B\langle X\rangle/(X\mathrm{Ha} - S_k^{p^{k-r}}) \to B\langle X\rangle/(X\mathrm{Ha} - S_k^{p^{k-r-1}})$$

where $B = R\hat{\otimes}_{\mathbb{Z}_p} C_{k,l}$ is as in Definition 6.2.1.5. Since E' reduces to $E^{(p)}$ mod π', it is clear that $\phi \bmod \pi'$ is the Frobenius $B \to B$ relative to $C_{k,l}$ when restricted to B. Since by construction, $\eta' \equiv \eta^p \bmod \pi'$, the map $\phi \bmod \pi'$ moreover sends $X \mapsto X^p$. We conclude that $\phi \bmod \pi'$ is the relative Frobenius as desired. That ϕ commutes with the transition maps in k and l is clear from the moduli description. This proves the Proposition for $k \geq r+1$.

Alternatively, for part 2, one could use [50] Theorem III.2.15 and argue by base-change to see that ϕ reduces to the relative Frobenius at all geometric points of $C_{k,l}$. □

We remark that since the generic fibre $\phi : \mathcal{X}_{r+1,k,l} \to \mathcal{X}_{r,k,l}$ commutes with the maps to $\mathcal{W}_{k,l}$, it is clear that it restricts for any $I \subseteq [l, \infty]$ to a map $\mathcal{X}_{r+1,k,I} \to \mathcal{X}_{r,k,I}$. By Lemma 6.2.4.1 and Lemma A.2.4, this has a canonical finite formal model

$$\mathfrak{X}_{r+1,k,I} \to \mathfrak{X}_{r,k,I}. \tag{25}$$

There is also a second natural map $\mathfrak{X}_{r',k,l} \to \mathfrak{X}_{r,k,l}$ which is a formal model of the natural inclusion $\mathcal{X}_{r+1,k,l} \hookrightarrow \mathcal{X}_{r,k,l}$. However, we shall not use this second map in the following.

Lemma 6.4.2. *Let $m \leq k \in \mathbb{Z}_{>0}$, $l \in p^{-m}\mathbb{Z}_{>0}, r \geq m$ and $n \leq r - m$. Then there is a natural map $\mathcal{I}G_{n,r+1,k,l} \to \mathcal{I}G_{n,r,k,l}$ that fits into a Cartesian diagram*

$$\begin{tikzcd} \mathcal{I}G_{n,r+1,k,l} \arrow[r, dotted, "\phi"] \arrow[d] & \mathcal{I}G_{n,r,k,l} \arrow[d] \\ \mathcal{X}_{r+1,k,l} \arrow[r, "\phi"] & \mathcal{X}_{r,k,l} \end{tikzcd}$$

and has a finite formal model $\mathfrak{IG}_{n,r+1,k,l} \to \mathfrak{IG}_{n,r,k,l}$.

Proof. Let us omit the indices k and l for the proof since they will not change. Recall that $\mathfrak{E}_{n,r}$ pulls back along $\phi_r : \mathcal{X}_{r+1} \to \mathcal{X}_r$ to $\mathfrak{E}_{r+1}/H_{1,r+1}$. By compatibility of the canonical subgroup with base change, $H_{n,r}$ therefore pulls back to the canonical subgroup $H_n(\mathfrak{E}_{r+1}/H_{1,r+1})$. The dual isogeny $\mathfrak{E}_{r+1}/H_{1,r+1} \to \mathfrak{E}_{r+1}$ on $\mathcal{X}_{r+1}$ now identifies $H_n(\mathfrak{E}_{r+1}/H_{1,r+1})^\vee = H_n(\mathfrak{E}_{r+1})^\vee$. We see from this that we have a Cartesian diagram

$$\begin{tikzcd} H^\vee_{n,r+1} \arrow[r, dotted, "\phi"] \arrow[d] & H^\vee_{n,r} \arrow[d] \\ \mathcal{X}_{r+1} \arrow[r, "\phi"] & \mathcal{X}_r. \end{tikzcd}$$

Thus the space which relatively over $\mathcal{X}_r$ parametrises isomorphisms $\underline{\mathbb{Z}/p^n\mathbb{Z}} \to H^\vee_{n,r}$ pulls back along ϕ to the space relatively over $\mathcal{X}_{r+1}$ parametrising isomorphisms $\underline{\mathbb{Z}/p^n\mathbb{Z}} \to H^\vee_{n,r+1}$.

The formal model as usual arises from normalisation, and is finite by excellence. □

6.5 Infinite level

At this point we can finally define the infinite level version of the Igusa tower:

Definition 6.5.1. Let $m \in \mathbb{Z}_{\geq 0}$, $l \leq l' \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$, $I = [l, l']$. In the following, all limits are taken in the category of S_m-adic formal schemes over $\mathfrak{W}_{k,I}$.

1. For $r \geq m$, we let $\mathfrak{X}_{r,\infty,I} := \varprojlim_{k\to\infty} \mathfrak{X}_{r,k,I}$ over $\mathfrak{W}_{k,I}$. For any $n \leq r - m$, we define
$$\mathfrak{IG}_{n,r,\infty,I} := \varprojlim_{k\to\infty} \mathcal{I}G_{n,r,k,I} \to \mathfrak{X}_{r,\infty,I}.$$
2. Let now $m \leq k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, i.e. we allow $k = \infty$. We set $\mathfrak{X}_{\infty,k,I} := \varprojlim_{r\to\infty} \mathfrak{X}_{r,k,I}$ where the transition maps are the Frobenius lifts. For any $n \in \mathbb{Z}_{>0}$, we similarly define
$$\mathfrak{IG}_{n,\infty,k,I} := \varprojlim_{r\to\infty} \mathfrak{IG}_{n,r,k,I} \to \mathfrak{X}_{\infty,k,I}.$$
3. Finally, for any $m \leq k \in \mathbb{Z}_{>0} \cup \{\infty\}$ we set $\mathfrak{IG}_{\infty,\infty,k,I} = \varprojlim_{n\to\infty} \mathfrak{IG}_{n,\infty,k,I}$.

Like before, we set $\mathfrak{IG}_{n,r,k,l} := \mathfrak{IG}_{n,r,k,[l,\infty]}$ also for n, r, k infinite. This is the weight space interval we will work with almost exclusively. It is clear that the transition maps in n, r, k, l extend to give commuting transition maps also at infinite level.

Remark 6.5.2. We note in contrast to the other indexes, the formal scheme $\mathfrak{IG}_{n,r,k,\infty}$ is not defined as some limit for $l \to \infty$, but instead simply refers to the case of $I = [\infty, \infty]$, i.e. the boundary of weight space. The link to the setting of perfectoid t-adic modular forms over $\mathbb{F}_p[[t^{1/p^\infty}]]$ is that we have $\mathfrak{X}'(1)^{\mathrm{perf}} = \mathfrak{X}_{\infty,\infty,\infty}$ and $\mathfrak{X}'_{\mathrm{Ig}(p^\infty)}(1)^{\mathrm{perf}} = \mathfrak{IG}_{\infty,\infty,\infty,\infty}$.

From now on, we work with the following setup, in which we allow n, r, k to be infinite:

Assumption 6.5.3. We let m, l, k, r, n be such that $m \in \mathbb{Z}_{\geq 0}$, $l \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$, $m \leq k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $m \leq r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $n \leq r - m$. Let $I \subseteq [l, \infty]$.

We let l', k', r', n' be such that $l \leq l' \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$, $k \leq k' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $r \leq r' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $n \leq n' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $n' \leq r' - m$.

For the rest of this section, we fix any $m, l, k, r, n, l', k', r', n'$ like in Assumption 6.5.3.

Lemma 6.5.4. *The subsheaf* $\mathcal{O}_{\mathfrak{X}_{r,k,l}} \subseteq \mathcal{O}_{\mathfrak{X}_{r,k,l}}[1/S]$ *on* $\mathfrak{X}_{r,k,l}$ *is integrally closed. Similarly, the subsheaf* $\mathcal{O}_{\mathfrak{IG}_{n,r,k,l}} \subseteq \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}[1/S]$ *on* $\mathfrak{IG}_{n,r,k,l}$ *is integrally closed.*

Proof. For $n, r, k < \infty$ this holds by Lemma 6.2.4 for $\mathfrak{X}_{r,k,l}$ and by definition for $\mathfrak{IG}_{n,r,k,l}$. The general case follows in the limit using Lemma A.2.2.1 □

Lemma 6.5.5. *$\mathfrak{X}_{r,k,l'} \to \mathfrak{W}_{k,l'}$ is the base change of $\mathfrak{X}_{r,k,l} \to \mathfrak{W}_{k,l}$ along $\mathfrak{W}_{k,l'} \to \mathfrak{W}_{k,l}$.*

Proof. For $k, r < \infty$, this is Lemma 6.2.4.3.(ii). The general case follows from forming the respective tower of Cartesian diagrams for the inverse limit $k \to \infty$ or $r \to \infty$. $\square$

Lemma 6.5.6. *For $k \geq d \in \mathbb{Z}_{\geq 0}$, we have a $C_{k,l} \xrightarrow{\sim} C_{k-d,lp^d}$-linear rescaling isomorphism*

$$r_d : \mathfrak{IG}_{n,r,k,l} \xrightarrow{\sim} \mathfrak{IG}_{n,r-d,k-d,lp^d}$$

which commutes with transition maps. In particular, there is a $\mathbb{Z}_p^\times$-linear rescaling map

$$r_k : \mathfrak{IG}_{\infty,\infty,\infty,l} \xrightarrow{\sim} \mathfrak{IG}_{\infty,\infty,\infty,lp^d}.$$

Proof. For $r, k < \infty$, we have a rescaling map $r_d : \mathfrak{X}_{r,k,l} \xrightarrow{\sim} \mathfrak{X}_{r-d,k-d,lp^d}$ by Lemma 6.2.2. On the generic fibre, this induces a $\mathbb{Z}/p^n\mathbb{Z}$-linear isomorphism $r_d : \mathcal{IG}_{n,r,k,l} \xrightarrow{\sim} \mathcal{IG}_{n,r-d,k-d,lp^d}$ since r_d commutes with the "arithmetic structure maps" to $\mathfrak{X}_{\mathbb{Z}_p}$ and thus identifies canonical subgroups. This gives the isomorphism $r_d : \mathfrak{IG}_{n,r,k,l} \xrightarrow{\sim} \mathfrak{IG}_{n,r-d,k-d,lp^d}$ upon normalisation.

The case of $r, k, n = \infty$ follows in the limit as r_d commutes with all transition maps. $\square$

Lemma 6.5.7. *For any $d \in \mathbb{N}$, the map $\mathcal{O}_{\mathfrak{X}_{r,k,I}}/S_k^d \hookrightarrow \mathcal{O}_{\mathfrak{IG}_{n,r',k',I}}/S_k^d$ is injective.*

Proof. For $n, r', k' < \infty$, this holds since $\mathfrak{IG}_{n',r',k',I} \to \mathfrak{X}_{r,k,I}$ is finite and $\mathfrak{X}_{r,k,I}$ is normal. The general case follows in the direct limit $r \to \infty$ and/or $k \to \infty$ and/or $n \to \infty$. $\square$

7 Galois invariants and canonical lifts

7.1 Galois invariants

Throughout this section, let us fix once and for all m, l, k, r, n, n' like in Assumption 6.5.3. When $n, r, k < \infty$, the morphism $\mathcal{IG}_{n,r,k,l} \to \mathcal{X}_{r,k,l}$ is a finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor. By normalisation, it induces a $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action also on $\mathfrak{IG}_{n,r,k,l} \to \mathfrak{X}_{r,k,l}$. Since this is clearly compatible with the transition maps in r, k, in the limit this gives rise to an $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action also in the case of general $r = \infty$ and/or $k = \infty$. For $r = \infty$, we can moreover consider the limit $n \to \infty$ for which we obtain a $\mathbb{Z}_p^\times$-action on $\mathfrak{IG}_{\infty,\infty,k,l}$. Roughly following §6.4 of [3] with our modified setup, the goal of this subsection is to prove the following:

Proposition 7.1.1. *Let l, k be like in Assumption 6.5.3. Then $\mathfrak{IG}_{\infty,\infty,k,l} \to \mathfrak{X}_{\infty,k,l}$ induces*

$$(\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,k,l}})^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}_{\infty,k,l}}.$$

The version of $k = 0$ is Proposition 6.3. of [3]. We are most interested in the case of $k = \infty$, which can be proved in a similar way, as we shall now discuss. On the way, we prove some related Lemmas that we only need in the next section.

Notation 7.1.2. To keep notation light, we use throughout this section the notation from Definition 6.2.1.5 and Notation 6.3.4: Let $r_0 \leq r$ and $k_0 \leq k$, then for some affine open $U \subseteq \mathfrak{X}_{r_0,k_0,l|U}$, let $A_{r,k,l} = \mathcal{O}_{\mathfrak{X}_{r,k,l}}(U)$ and $B_{n,r,k,l} = \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}(U)$.

Proposition 7.1.3. *For $n < \infty$, the morphism $\mathfrak{IG}_{n,r,k,l} \to \mathfrak{X}_{r,k,l}$ is integral.*

Since $\mathfrak{IG}_{n,r,k,l}$ is integrally closed in $\mathcal{IG}_{n,r,k,l}$ by Lemma 6.5.4, we may a posteriori regard $\mathfrak{IG}_{n,r,k,l} \to \mathfrak{X}_{r,k,l}$ as being the "normalisation" of $\mathfrak{X}_{r,k,l}$ in $\mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}[1/S]$, even for $r, k = \infty$.

Proof. Let us abbreviate $B_\infty := B_{n,r,k,l}$ and $B_i := B_{n,r_i,k_i,l}$ for some sequences of $r_i \leq r$, $k_i \leq k$, $r_i, k_i < \infty$ and $r_i \to r, k_i \to k$. Similarly $A_\infty = A_{r,k,l}$ and $A_i = A_{r_i,k_i,l}$. We then have $B_\infty = (\varinjlim_i B_i)^\wedge$.

Since $B_i[1/S]$ is Galois over $A_i[1/S]$ with group $G := \mathbb{Z}/p^n\mathbb{Z}$, and B_i is the integral closure of A_i in $B[1/S]$, we have $B_i^G = A_i$ for $i < \infty$. In particular, for any $b_i \in B_i$, the polynomial $\prod_{\sigma \in G}(X - \sigma(b_i))$ has coefficients in A_i.

We claim that A_∞ is topologically closed in B_∞. Indeed, by Lemma 6.5.7, the reduction $A_\infty/S_m^d \hookrightarrow B_\infty/S_m^d$ is injective for any $d \in \mathbb{N}$. This shows that any sequence in A_∞ that is convergent in B_∞ is also convergent in A_∞. Since A_∞ and B_∞ are complete and separated and $A_\infty \to B_\infty$ is continuous, this shows that A_∞ is closed in B_∞.

Let now $b \in B_\infty$ and let $b_i \in B_{n,r_i,k_i,l}$ with $b_i \to b$ inside $B_{n,r,k,l}$ for $i \to \infty$. We claim that the polynomial $f_b := \prod_{\sigma \in G}(X - \sigma(b)) \in B_\infty[X]$ has coefficients in A_∞. Indeed, since σ is continuous, the coefficients of $f_{b_i} = \prod_{\sigma \in G}(X - \sigma(b_i))$ converge to the coefficients of f_b in B_∞. But f_{b_i} has coefficients in $B_i^G = A_i$ and thus in A_∞. The polynomial f_b thus has coefficients in the topological closure of A_∞ in B_∞, which as we have just seen is A_∞. This shows that b is integral over A_∞ as desired. □

Assumption 7.1.4. From now on until the next section, we will for simplicity restrict attention to the case $r + 1 \geq k$. This restriction is not essential and could be avoided, but this would require some additional steps in our Noetherian approximation arguments. The condition becomes vacuous in the case $r = \infty$ which we are ultimately mainly interested in.

Lemma 7.1.5. *Assume $k \leq r + 1 < \infty$. Then $S_k \in \mathrm{Hdg}_{r,k,l}$.*

Proof. It suffices to prove this locally on some cover. Let $\mathrm{Spf}(R) = U \subseteq \mathfrak{X}_{k,l}$ be an affine open on which ω is trivial and choose a trivialisation of ω. We can then write $\mathrm{Ha} \in R$. By the definition of $\mathrm{Hdg}_{r,k,l}$ in Definition 6.3.1, we have $\mathrm{Hdg}_{r,k,l} = (p, \mathrm{Ha}) \subseteq \mathcal{O}_{\mathfrak{X}_{r,k,l}}(U)$.

If $k \leq r + 1$, then $\mathcal{O}_{\mathfrak{X}_{r,k,l}}(U) = R\langle X\rangle/(X\mathrm{Ha}^{p^{r+1-k}} - S_k)$, thus $S_k \in \mathrm{Hdg}_{r,k,l}$. □

Proposition 7.1.6. *Assume $n, r, k < \infty$ and $k \leq r + 1$. Then for each affine $U \subseteq \mathfrak{X}_{r,k,l}$, the $\mathcal{O}_{\mathfrak{X}_{r,k,l}}(U)$-module $H^1((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}(U))$ is annihilated by $S_k^{p^n}$.*

Proof. Let us write $A = \mathcal{O}_{\mathfrak{X}_{r,k,l}}(U)$ and $B_n := \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}(U)$. Then by Lemma 7.1.5 we have $S_k \in \mathrm{Hdg}_{r,k,l}$. By [3] Proposition 3.5, we have $\mathrm{Hdg}_{r,k,l}^{p^{n-1}} \subseteq \mathrm{Tr}_{B_n|B_{n-1}}(B_{n+1})$ for $k = 0$ and $n \geq 2$. We conclude that the same also holds for general $k \leq r + 1$: This follows from rescaling by $r_k : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r-k,0,lp^k}$ which identifies Hodge ideals since these come from pullback of $\mathfrak{X}_{\mathbb{Z}_p}$. Combinining these observations, we have $S_k^{p^{n-1}} \in \mathrm{Tr}_{B_n|B_{n-1}}(B_{n+1})$. Using that $\mathrm{Tr}_{B_n|A} = \mathrm{Tr}_{B_1|A} \circ \cdots \circ \mathrm{Tr}_{B_{n-1}|B_{n-2}} \circ \mathrm{Tr}_{B_n|B_{n-1}}$, this shows inductively that

$$(S_k^{p^n}) \subseteq (S_k^{(p^n-1)/(p-1)}) \subseteq \mathrm{Tr}_{B_n|A}(B_n).$$

By [56], Corollary 3.2.1, this implies that $S_k^{p^n}$ annihilates $H^1(G, B_n)$. □

Corollary 7.1.7. *Assume $n < \infty$, and either $k \leq r + 1$ or $r, k < \infty$. Then we have*

$$(\mathcal{O}_{\mathfrak{IG}_{n,r,k,l}})^{(\mathbb{Z}/p^n\mathbb{Z})^\times} = \mathcal{O}_{\mathfrak{X}_{r,k,l}}.$$

Proof. For $r, k < \infty$ this holds true (also if $k \geq r + 1$) on the generic fibre since the map $\mathcal{IG}_{n,r,k,l} \to \mathcal{X}_{r,k,l}$ is a finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor and $\mathfrak{IG}_{n,r,k,l}$ is the normalisation.

We now consider the limit $r \to \infty$, and either $k < \infty$ constant or $r + 1 \geq k \to \infty$. In this projective system, the opens obtained from pullback of affine opens U along $\mathfrak{X}_{\infty,k,l} \to \mathfrak{X}_{r,k_0,l}$ for $r \to \infty$ and $k \geq k_0 < \infty$ form a basis for the topology. Since by Proposition 7.1.6, the group cohomology $H^1((\mathbb{Z}/p^n\mathbb{Z})^\times, \varinjlim_{k+1 \leq r \to \infty} \mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}(U))$ has uniformly bounded S_k-torsion, we may commute invariants and completion by Lemma A.3.11.2. □

Corollary 7.1.8. *Assume $n < \infty$, and $r+1 \geq k$ or $r, k < \infty$. Then $b \in B_{n,r,k,l}[1/S]$ is in $B_{n,r,k,l}$ if and only if the polynomial $f_b = \prod_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times}(X - \sigma(b))$ has coefficients in $A_{r,k,l}$.*

Proof. If f_b has coefficients in $A_{r,k,l}$, then clearly b is integral. Conversely, if b is integral, then so are the $\sigma(b)$. The coefficients of f_b are then in $B^G_{n,r,k,l}$ for $G = (\mathbb{Z}/p^n\mathbb{Z})^\times$, which equals $A_{r,k,l}$ by Corollary 7.1.7. □

proof of Proposition 7.1.1. The case of $n < \infty$ is Corollary 7.1.7. We are left to consider the case of $n = \infty$, which by Assumption 6.5.3 implies $r = \infty$.

The proof relies on a refinement of Proposition 7.1.6: As explained in the proof of [3], Proposition 6.3, one can use a comparison of Hodge ideals in the limit $r \to \infty$ to show that for $r = \infty$, the group cohomology $H^1((\mathbb{Z}/p^n\mathbb{Z})^\times, B_{n,\infty,0,lp^k})$ is already annihilated by T for all $n < \infty$. Via the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant rescaling morphism $B_{n,\infty,k,l} \xrightarrow{\sim} B_{n,\infty,0,lp^k}$, we conclude that $H^1((\mathbb{Z}/p^n)^\times, B_{n,\infty,k,l})$ is annihilated by S_k. The same is therefore true for

$$H^1(\mathbb{Z}_p^\times, \varinjlim_n B_{n,\infty,k,l}) = \varinjlim_n H^1((\mathbb{Z}/p^n\mathbb{Z})^\times, B_{n,\infty,k,l}),$$

where the equality is a fact in profinite group cohomology, see [45], Proposition 1.5. Since in particular the S-torsion in $H^1(\mathbb{Z}_p^\times, \varinjlim_n B_{n,\infty,k,l})$ is bounded, it follows from Lemma A.3.11.2 that we may commute S-adic completion and $\mathbb{Z}_p^\times$-invariants:

$$(B_{\infty,\infty,k,l})^{\mathbb{Z}_p^\times} = ((\varinjlim_n B_{n,\infty,k,l})^\wedge)^{\mathbb{Z}_p^\times} \overset{\text{A.3.11.2}}{=} ((\varinjlim_n B_{n,\infty,k,l})^{\mathbb{Z}_p^\times})^\wedge \overset{\text{Cor. 7.1.7}}{=} (A_{\infty,k,l})^\wedge = A_{\infty,k,l}.$$

This shows that the Proposition holds on a basis, and thus everywhere. □

7.2 Reduction versus normalisation

Throughout we fix m, l, l', k, r, n as in Assumption 6.5.3 and assume for simplicity that $r+1 \geq k$. We moreover set $\pi := p/S_m^{lp^m}$. We have seen in §6.4 that there is a natural map

$$\varphi : \mathfrak{IG}_{n,r,k,l'} \to \mathfrak{IG}_{n,r,k,l} \times_{\mathfrak{W}_{k,l}} \mathfrak{W}_{k,l'}$$

and one may ask when this is an isomorphism. The goal of this section is to prove that this is the case for $l' = \infty$ and r, k large enough, and in particular for $r = k = \infty$.

To unravel what the question is asking for, we note that $\mathfrak{W}_{k,\infty} \to \mathfrak{W}_{k,l}$ is defined by

$$\mathbb{Z}_p[[T]][(1+T)^{1/p^k}]\langle p/S_m^{lp^m}\rangle \to \mathbb{F}_p[[T^{1/p^k}]]$$

which can be described as reduction mod π. The map φ is therefore locally of the form

$$B_{n,r,k,l}/\pi \to B_{n,r,k,\infty}.$$

We start by checking that this map is an injection.

Lemma 7.2.1. *Let A be any ring with non-zero divisors $\pi, t \in A$ such that t is a non-zero divisor on $A/(\pi)$. Let B be any A-algebra on which t is a non-zero divisor and suppose B is integrally closed in $B[1/t]$. Suppose further that there is a finite group G that acts A-linearly on B with $B^G = A$. Then t is a non-zero divisor on $B/(\pi)$.*

Proof. We first note that the assumption on t being a non-zerodivisor mod π is equivalent to $A/(\pi) \to A[1/t]/(\pi)$ being injective. By induction, it is also equivalent to saying that for any $n \in \mathbb{N}$, $A \cap \pi^n A[1/t] = \pi^n A$. The same argument shows that it suffices to prove that $B/(\pi) \to B[1/t]/(\pi)$ is injective.

Let $x \in B$ with $x \in \pi B[1/t]$, say $x = \pi y$ for some $y \in B[1/t]$. We want to show $x \in \pi B$. The polynomial $f_x = \prod_{\sigma \in G}(X - \sigma(x)) = X^m + a_{n-1}X^{n-1} + \cdots + a_0$ has coefficients in

$B^G = A$. Since $\sigma(x) \in \pi B[1/t]$, for any $0 \leq i \leq n-1$, the coefficient a_i is moreover in $(\pi^{n-i}B[1/t])^G = \pi^{n-i}A[1/t]$. It is thus in $A \cap \pi^{n-i}A[1/t] = \pi^{n-i}A$. We can thus write $a_i = \pi^{n-i}a_i'$ for some $a_i' \in A$ for all $0 \leq i \leq n-1$. But then $\prod_{\sigma\in}(X - \sigma(y)) = X^m + a_{n-1}'X^{n-1} + \ldots + a_0'$ since π is a non-zero-divisor. This shows that y is integral over A. Since B is integrally closed in $B[1/t]$, this shows $y \in B$, thus $x \in \pi B$ as desired. $\square$

Lemma 7.2.2. *The element S_m is a non-zero divisor on $\mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}/\pi$.*

Proof. We may assume $l < \infty$, since $\pi = 0$ if $l = \infty$. We first note that $\mathcal{O}_{\mathfrak{X}_{r,k,l}}/\pi = \mathcal{O}_{\mathfrak{X}_{r,k,\infty}}$ by Lemma 6.5.5. Then in case that $n < \infty$, Lemma 7.2.1 applies for $t = S_m$ and shows that S_m is a non-zero-divisor on $B_{n,r,k,l}/\pi$. Equivalently by a Snake lemma diagram, π is a non-zero-divisor on $B_{n,r,k,l}/S_m$. The general case follows by taking the direct limit $n \to \infty$ of the maps $B_{n,r,k,l}/S_m \xhookrightarrow{\cdot\pi} B_{n,r,k,l}/S_m$ which shows that π is a non-zero-divisor on $B_{n,r,k,l}/S_m$ in general. Equivalently, S_m is a non-zero-divisor on $B_{n,r,k,l}/\pi$. $\square$

Lemma 7.2.3. *Assume $n, r, k < \infty$. Then the natural morphism $\mathcal{O}_{\mathfrak{IG}_{n,r,k,l}}/\pi \hookrightarrow \mathcal{O}_{\mathfrak{IG}_{n,r,k,\infty}}$ is injective and becomes an isomorphism upon inverting S_m or, in this case equivalently, T.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} B_{n,r,k,l}/\pi \arrow[r] \arrow[d] & B_{n,r,k,l}/\pi[1/S_m] \arrow[d] \\ B_{n,r,k,\infty} \arrow[r] & B_{n,r,k,\infty}[1/T] \end{tikzcd} \tag{26}$$

The morphism in the top row is injective by Lemma 7.2.2. The morphism on the right is an isomorphism by the Cartesian diagram in Lemma 6.3.6 applied to $k' = k$, $l = l$, $l' = \infty$. Thus the diagonal morphism is injective, and thus so is the map on the left. $\square$

The Lemma shows that for $n, r, k < \infty$ we can regard both $B_{n,r,k,l}/\pi$ and $B_{n,r,k,\infty}$ as submodules of $B_{n,r,k,\infty}[1/T]$. In light of diagram (26), the question of when φ is an isomorphism therefore amounts to asking when $B_{n,r,k,l}/\pi$ is already integrally closed in $B_{n,r,k,\infty}[1/T]$, or, in other words, when normalisation commutes with reduction mod π. While of course this is not always the case, the following key lemma says that in the present situation this is true for r, k large enough.

Lemma 7.2.4. *Let $n, r, k < \infty$, then for $0 \ll d < \infty$ the map $\mathcal{O}_{\mathfrak{IG}_{n,r+d,k+d,l}} \to \mathcal{O}_{\mathfrak{IG}_{n,r+d,k+d,\infty}}$ is surjective, and therefore $\mathfrak{IG}_{n,r+d,k+d,\infty} \hookrightarrow \mathfrak{IG}_{n,r+d,k+d,l}$ is a closed immersion.*

Proof. We first prove the following statement: For $r, k < \infty$ and $n \leq r - m$, the morphism

$$\mathcal{O}_{\mathfrak{IG}_{n,r,k,l}} \to \mathcal{O}_{\mathfrak{IG}_{n,r,k,\infty}}$$

is surjective for $l \in p^{-m}\mathbb{Z}_{\geq 0}$ with $l \gg 0$. This implies the Lemma by rescaling: For $l < \infty$ and $n \in \mathbb{Z}_{\geq 0}$, by Lemma 6.5.6, the rescaling isomorphism r_d gives a commutative diagram

$$\begin{tikzcd} B_{n,r+d,k+d,l} \arrow[r] \arrow[d, "r_d", "\wr"'] & B_{n,r+d,k+d,\infty} \arrow[d, "r_d", "\wr"'] \\ B_{n,r,k,lp^d} \arrow[r] & B_{n,r,k,\infty}. \end{tikzcd}$$

By the first part of the Lemma, the bottom row becomes an isomorphism for lp^d large enough, and thus for d large enough. Consequently, the same is true for the top row.

Let us drop r, k, n from notation since these won't change in the following, that is we write $A_l := A_{r,k,l}$ and $A_\infty = A_{r,k,\infty}$, etc.

Due to the finiteness assumption on r and k, the ring extension $A_\infty \to B_\infty$ is finite. Let $x_1, \ldots, x_j$ be a set of generators of B_∞ as an A_∞-module. By Lemma 7.2.3, the inclusion

$B_l/\pi \hookrightarrow B_\infty$ becomes an equality after inverting S_m. Therefore we can find lifts $\tilde{x}_1, \dots, \tilde{x}_j \in B_l[1/S]$ of the $x_1, \dots, x_j$. We need to prove that for $l' \gg l$ large enough, the images of the $\tilde{x}_1, \dots, \tilde{x}_j$ under the natural map $B_l[1/S] \to B_{l'}[1/S]$ are already contained in $B_{l'}$.

It suffices to show this for one generator, say $x := x_i$. Set $G = (\mathbb{Z}/p^n\mathbb{Z})^\times$, $N := |G|$, then the polynomial $f_{\tilde{x}} = \prod_{\sigma \in G}(X - \sigma(\tilde{x})) = X^N + a_{N-1}X^{N-1} + \dots + a_0$ has coefficients $a_i \in B_l[1/S]^G = A_l[1/S]$. We now chase the images of the a_i in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi A_l \arrow[r] \arrow[d] & A_l \arrow[r] \arrow[d] & A_\infty \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & pA_l[1/S] \arrow[r] & A_l[1/S] \arrow[r] & A_\infty[1/S] \arrow[r] & 0.
\end{tikzcd}$$

Since $B_l \to B_\infty$ is G-equivariant, the polynomial reduces mod π to $f_x = \prod_{\sigma \in G}(X - \sigma(x)) = X^N + \overline{a}_{N-1}X^{N-1} + \dots + \overline{a}_0$ where $\overline{a}_i$ is the image of a_i in $A_\infty[1/S]$. Since x is integral, Lemma 7.1.8 says that $\overline{a}_i$ is already in A_∞. If now a'_i is any lift of $\overline{a}_i$ to A_l, then the exactness of the bottom row in the diagram shows $\delta_i := a_i - a'_i \in pA_l[1/S]$. We can therefore find $l \leq l' \in \mathbb{Z}_{\geq 0}$ large enough such that $S^{l'}\delta_i \in pA_l$ for all i (recall that $S^{l'}$ is only defined up to a unit, but this statement makes sense regardless). Equivalently, $\delta_i \in (p/S^{l'})A_l$.

But since by definition $p/S^{l'} \in A_{l'} := A_{r,k,l'}$, this implies that $A_l[1/S] \to A_{l'}[1/S]$ sends each δ_i into $A_{l'}$. Since also $a'_i \in A_{l'}$, we have $a_i = \delta_i + a'_i \in A_{l'}$. Thus f_x has coefficients in $A_{l'}$ which again by Lemma 7.1.8 implies that $x \in B_{l'}$. This proves the claimed statement. □

Proposition 7.2.5. *Let $l \in \mathbb{Z}[1/p]_{>0}$ and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then the natural morphism $i : \mathfrak{IG}_{n,\infty,\infty,\infty} \hookrightarrow \mathfrak{IG}_{n,\infty,\infty,l}$ is a closed immersion defined by $\pi = p/S^l$. In particular, the following sequence of sheaves on $\mathfrak{IG}_{n,\infty,\infty,l}$ is exact:*

$$0 \to \pi\mathcal{O}_{\mathfrak{IG}_{n,\infty,\infty,l}} \to \mathcal{O}_{\mathfrak{IG}_{n,\infty,\infty,l}} \to i_*\mathcal{O}_{\mathfrak{IG}_{n,\infty,\infty,\infty}} \to 0.$$

Proof. This can be seen locally, so it suffices to prove that the natural morphism

$$B_{n,\infty,\infty,l}/\pi \to B_{n,\infty,\infty,\infty}.$$

is an isomorphism. By Lemma 7.2.3, the natural morphism

$$B_{n,r+d,d,l}/\pi \to B_{n,r+d,d,\infty}$$

is injective for any $n, r, d < \infty$ with $d \geq m$. By Lemma 7.2.4, it is also surjective for d large enough. Consequently, it is then an isomorphism. This remains true after taking the direct limit over $k \to \infty$, and after passing to completions, where we use Lemma 7.2.2 to commute completion and reduction mod π. This shows the result for $n < \infty$. The same limit argument applies for the direct limit over n, which shows the result in the case $n = \infty$. □

7.3 Canonical lifts of Igusa curves

In this section, we relate the canonical lift of the perfectoid Igusa scheme $\mathfrak{IG}_{\infty,\infty,\infty,\infty}$ over $\mathbb{F}_p[[t^{1/p^\infty}]]$ to the integral family of perfectoid modular curves $\mathfrak{IG}_{\infty,\infty,\infty,l}$ for any $l \in \mathbb{Z}[1/p]$.

We recall that for the sake of overall notational consistency we denote by t the image of $T \in \mathbb{Z}_p[[T]]$ under the reduction mod p. As discussed in §6.1, there is a canonical isomorphism $W(\mathbb{F}_p[[t^{1/p^\infty}]]) = \mathbb{Z}_p[[T]]\langle(1+T)^{1/p^\infty}\rangle$ which identifies the perfected weight space $\mathfrak{W}_\infty$ with the canonical Witt lift of the point $\mathrm{Spf}(\mathbb{F}_p[[t^{1/p^\infty}]])$ that the modular curves in characteristic p are defined over. The goal of this section is to show that a similar identification is possible for the whole overconvergent Igusa tower of modular curves.

Remark 7.3.1. This is conceptually very closely related to the modular curve case of the more general canonical lift discussed in [16], §4.3 which can be described in terms of fibres of strata under the Hodge–Tate period map. The differences between the two settings are:

- Our Igusa space is a formal scheme over $\mathbb{F}_p[[t^{1/p^\infty}]]$ and lifts to $W(\mathbb{F}_p[[t^{1/p^\infty}]])$, whereas the Igusa scheme in [16] is a scheme over $\overline{\mathbb{F}}_p$ and lifts to $W(\overline{\mathbb{F}}_p)$.
- We use a slightly different notion of "Igusa structures": While we follow [36] in calling Igusa moduli problem the trivialisation of $\ker V^n$, Caraiani–Scholze in [16] follow Mantovan's convention which also trivialises other subquotients of the p-divisible group. But over the generically ordinary locus, the two are very closely related: Since in the ordinary case, the two graded pieces of the slope filtration are dual, the pro-Igusa variety $\mathscr{I}^{\mathrm{ord}}_{\mathrm{Mant}}$ of [16], Definition 4.3.6, can in the case of the modular curve be described as the scheme $X^{\mathrm{ord}}_{\mathbb{F}_p,\mathrm{Ig}(p^\infty)} \times_{X^{\mathrm{ord}}_{\mathbb{F}_p}} X^{\mathrm{ord}}_{\mathbb{F}_p,\mathrm{Ig}(p^\infty)}$ after removing the cusps of the latter.
- Since we consider a neighbourhood of the ordinary locus, our version of the canonical lift could be thought of as an overconvergent version of the very special case of the ordinary locus of the modular curve of the setting in [16].

We begin by formalising what we mean by canonical lifts to characteristic 0 in the t-adic setting. We do this by the following Lemma, whose proof is a straightforward computation.

Lemma 7.3.2. *Let $\mathfrak{Y} \to \mathrm{Spf}(\mathbb{F}_p[[t^{1/p^\infty}]])$ be a perfect t-torsionfree formal scheme. Then one can associate to $\mathfrak{Y}$ a canonical lift $W\mathfrak{Y} \to \mathrm{Spf}(\mathbb{Z}_p[[T]]\langle(1+T)^{1/p^\infty}\rangle) = \mathfrak{W}_\infty$ which locally on an affine open $\mathrm{Spf}(A) \subseteq \mathfrak{Y}$ is given by $\mathrm{Spf}(W(A))$ where $W(A)$ is endowed with the (p,T)-adic topology. This construction is functorial.*

In the language of the Lemma, the goal of this section is to construct a natural morphism $\mathfrak{IG}_{\infty,\infty,\infty,l} \to W\mathfrak{IG}_{\infty,\infty,\infty,\infty}$. In the case of $\epsilon = 0$ and $l = 0$ (which we do not consider here) this would be an isomorphism, but this is not true in general: The left hand side is T-adic, while the right hand side is only (p,T)-adic. For the construction, we therefore need the following lifting property of the Witt vectors, which is perhaps slightly non-standard.

Our motivation is the following: Say we have a perfect $\mathbb{F}_p$-algebra A and a p-adically complete algebra V with perfect residue ring V/p, then any morphism $\varphi : A \to V/p$ lifts to $W(A) \to V$. For any $t \in V$, we obtain by composition a map $\Phi : W(A) \to V \to R := V\langle p/t\rangle$ which reduces mod $\pi = p/t$ to $\varphi : A \to V/p = R/\pi$. Our goal is now to construct the map Φ from $\varphi : A \to R/\pi$ without reference to V. In our application, R will be $\mathcal{O}_{\mathfrak{IG}_{n,\infty,\infty,l}}$.

Lemma 7.3.3. *let A be a perfect $\mathbb{F}_p$-algebra endowed with the J-adic topology for some $J \subseteq A$. Let R be a $\mathbb{Z}_p$-algebra that is I-adically complete for some $I \subseteq R$. Suppose there is $\pi \in I$ such that $\pi | p$ in R. Then any continuous ring homomorphism $\varphi : A \to R/\pi$ lifts uniquely to a continuous ring homomorphism $[\varphi] : W(A) \to R$, where $W(A)$ is equipped with the $(p,[J])$-adic topology.*

Proof. We first note that there is a natural multiplicative morphism $[-]_\varphi : A \to R$ defined as usual: For any $a \in A$, we choose any preimage $z_n = z_n(a)$ of $\varphi(a^{1/p^n})$ under the projection $R \to R/\pi$. Then $z_{n+1}^p \equiv z_n \mod \pi$. The assumption that $\pi|p$, say $p = \pi t$, implies that

$$\text{for any } x, y \in R \text{ and } r \in \mathbb{N} \text{ with } x \equiv y \mod \pi^r, \text{ we have } x^p \equiv y^p \mod \pi^{r+1}. \qquad (27)$$

Indeed, if $y = x + h\pi^r$, then $y^p = x^p + p\pi^r(\dots) + (h\pi^r)^p = x^p + \pi^{r+1}t(\dots) + (h\pi^r)^p \equiv x^p \mod \pi^{r+1}$. Therefore, we have $z_{n+1}^{p^{n+1}} \equiv z_n^{p^n} \mod \pi^{n+1}$, and thus also $\mathrm{mod}\, I^{n+1}$. Since R is I-adically complete, this implies that $z_n^{p^n}$ converges to some $z \in R$ for $n \to \infty$ and we set $[x]_\varphi := z$. One verifies as usual that this does not depend on the choice of z_n and defines a multiplicative map $[-]_\varphi : A \to R$. We now define:

$$[\varphi] : W(A) \to R, \quad \sum [a_n]p^n \mapsto \sum [a_n]_\varphi p^n.$$

By the theory of Witt polynomials, this is in fact a ring homomorphism: We cannot apply the "usual" p-adic lifting property of the Witt vectors, since we do not assume that R is p-adically complete, but the usual proof that the map thus defined is a ring homomorphism using ghost components goes through using the coarser I-adic topology, because of equation (27).

This map is continuous: The continuity of $\varphi : A \to R/\pi$ means that there is $m \in \mathbb{N}$ such that $\varphi(J^m) \subseteq I$. Replacing J by J^m, we can without loss of generality assume that $m = 1$. Then the reduction of $[\varphi]([J])$ lands in $I \subseteq R/\pi$. Since $\pi \in I$, this shows $[\varphi]([J]) \subseteq I$. Since moreover $p \in I$, linearity implies that $[\varphi]((p, [J])) \subseteq I$. This shows that $[\varphi]$ is continuous.

Uniqueness follows as usual: By continuity and linearity it suffices to see that any other lift ϕ agrees with $[\varphi]$ on the elements $[a]$ for $a \in A$. Since $\phi \equiv [\varphi] \bmod \pi$, we have $\phi([a]) = \phi([a^{1/p^n}])^{p^n} \equiv [\varphi]([a^{1/p^n}])^{p^n} = [\varphi]([a]) \bmod I^{n+1}$ for all n by (27). Therefore $\phi = [\varphi]$. □

Using this generalised lifting property, we can finally prove the main result of this section:

Proposition 7.3.4. *For all $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $l \in \mathbb{Z}[1/p]_{>0}$, the map*

$$\mathfrak{IG}_{n,\infty,\infty,l}/(p/S^l) \xrightarrow{\sim} \mathfrak{IG}_{n,\infty,\infty,\infty}$$

from Proposition 7.2.5 lifts to a $\mathbb{Z}_p^\times$-equivariant morphism over perfected weight space

$$\mathfrak{IG}_{n,\infty,\infty,l} \to W\mathfrak{IG}_{n,\infty,\infty,\infty},$$

functorially in n and l. The functoriality in l uniquely characterises these lifts.

Proof. The idea of the proof is to apply the Lifting Lemma 7.3.3 to the isomorphism of Proposition 7.2.5 which is locally of the form $\psi_n : B_{n,\infty,\infty,\infty} \xrightarrow{\sim} B_{n,\infty,\infty,l}/(p/S^l)$ (here and in the following we use Notation 6.1.16). For this we set in the notation of the Lemma $A = B_{n,\infty,\infty,\infty}$, which is perfect and $J := (T)$-adic. We would like to set $R = B_{n,\infty,\infty,l}$ which is $I = (S)$-adically complete. We would moreover like to set π to p/S^l. However, this does not work for the Lemma since p/S^l is only powerbounded, not topologically nilpotent, and in particular p/S^l is not be contained in $I = (S)$ as required for the Lemma.

To motivate our way to solve this, we note that we have isomorphisms $B_{n,\infty,\infty,\infty} \xrightarrow{\sim} B_{n,\infty,\infty,l}/(p/S^l)$ for any $l \in \mathbb{Z}[1/p]_{>0}$, and these are compatible in l. Morally, this should in particular work for $l = 0$ – a case we did not consider. At least in the ordinary case, however, one can make this precise following [16] and actually identify "$\mathfrak{IG}_{n,\infty,\infty,0} = W\mathfrak{IG}_{n,\infty,\infty,\infty}$".

With this idea in mind, instead of setting $l = 0$, we choose any $0 < l_0 < l$ in $\mathbb{Z}[1/p]_{>0}$ and set $\pi := p/S^{l_0}$. Then $(\pi) = (p/S^{l_0}) = (S^{l-l_0} \cdot p/S^l)$ becomes topologically nilpotent in $B_{n,\infty,\infty,l}$. In particular, after setting $I = (S^{l-l_0})$, we indeed have $\pi \in I$. We clearly also still have $\pi | p$. Moreover, the natural map $B_{n,\infty,\infty,l_0} \to B_{n,\infty,\infty,l}$ induces a natural map

$$\varphi : B_{n,\infty,\infty,\infty} \xrightarrow[\sim]{\psi_{l_0}} B_{n,\infty,\infty,l_0}/(p/S^{l_0}) \to B_{n,\infty,\infty,l}/(p/S^{l_0})$$

We can now actually apply Lemma 7.3.3 to this morphism, and obtain the desired lift

$$[\varphi] : W(B_{n,\infty,\infty,\infty}) \to B_{n,\infty,\infty,l},$$

continuous for the (p, T)-adic topology on the left. Due to the functoriality of the Witt lift construction by Lemma 7.3.2, and the uniqueness assertion in Lemma 7.3.3, this construction glues, is independent of l and the auxiliaray choice of l_0, and is compatible for different l.

The assertion we are left to check is the uniqueness: For this let (Ψ_l) be any family of compatible lifts $\Psi_l : W(B_{n,\infty,\infty,\infty}) \to B_{n,\infty,\infty,l}$ of the maps (ψ_l). Then by the uniqueness assertion in Lemma 7.3.3, the lift Ψ_l is uniquely determined by its reduction mod p/S^{l_0}. On the other hand, the functoriality condition ensures that Ψ_l equals the composition of Ψ_{l_0} with the natural map $B_{n,\infty,\infty,l_0} \to B_{n,\infty,\infty,l}$. Since Ψ_{l_0} mod p/S^{l_0} equals ψ_{l_0} by assumption, this determines Ψ_l mod p/S_{l_0} uniquely to be the map φ we used for the construction. □

8 Traces and sousperfectoidness

The goal of this section is to lay the technical foundations for two tasks which are closely interconnected: The first is to obtain trace maps for modular curves over the perfectified weight space, which allow one to pass from perfectoid modular forms to true modular forms.

In order to be able to carry this out in families in regions close to the centre of weight space, we need to glue on the level of analytic adic spaces rather than on the formal schemes involved. For this to be well-defined, we need to know that the adic spaces $\mathcal{X}_{r,\infty,l}$ and $\mathcal{X}_{\infty,\infty,l}$ are sheafy. This is the second goal of this section. As usual in the non-Noetherian situation, we use the criterion of Buzzard–Verberkmoes [15] and Mihara [44] and show that these spaces are stably uniform, and we do so by using the concept of sousperfectoid adic spaces in the sense of [53], §6.3 due to Hansen–Kedlaya.

In order to prove that $\mathcal{X}_{\infty,\infty,l}$ and $\mathcal{X}_{r,\infty,l}$ are sousperfectoid, we first find an explicit perfectoid cover by a perfectoid space $\widetilde{\mathcal{X}}_{\infty,\infty,l}$: The idea is that $\mathfrak{X}_{\infty,\infty,l} \to \mathrm{Spf}(\mathbb{Z}_p)$ is relatively perfectoid, and therefore we should base-change to $\mathcal{O}_K$ for some perfectoid field extension K of $\mathbb{Q}_p$ to get an integrally perfectoid space. We note, however, that since $\mathcal{X}_{r,\infty,l}$ does not live over a field, we are not in the setting of [48] but instead need to use perfectoid spaces in the sense of [53]. There is therefore a subtlety that we need to take care of, namely we need to make sure that our spaces are still locally uniform after the base change to $\mathcal{O}_K$.

Once this is verified, we will obtain an integrally perfectoid cover $\widetilde{\mathfrak{X}}_{\infty,\infty,l}$ with maps

$$\widetilde{\mathfrak{X}}_{\infty,\infty,l} \to \mathfrak{X}_{\infty,\infty,l} \to \mathfrak{X}_{r,\infty,l}$$

that admit bounded generic sections arising from normalised trace maps. For the map on the right, this is known by an adaptation by Andreatta–Iovita–Pilloni of a lemma by Scholze.

8.1 Traces in the Frobenius tower

Notation 8.1.1. For a finite morphism of rings $\varphi : A \to B$ that is locally free of rank n, we denote its trace by Tr_φ, or simply by Tr if φ is clear from context. If moreover n is a non-zero-divisor on B, we denote by $\mathrm{tr}_\varphi = 1/n \cdot \mathrm{Tr}_\varphi : B \to A[1/n]$ the normalised trace.

Our next goal is to adapt the Tate traces of [3], Corollaire 6.1 to the setting of $k = \infty$. We note that the bounds we obtain are slightly different to those in [3], the reason being that Proposition 6.4.1.2 produces a different ideal modulo which ϕ equals the Frobenius. In particular, our bounds are not optimal, but they suffice for our later applications.

Proposition 8.1.2 (cf [3], Corollaire 6.1). *For any $m+2 \leq r \leq r' \in \mathbb{Z}_{\geq 0}$ and $l \in 1/p^m\mathbb{Z}_{>0}$, the trace of $\phi : \mathfrak{X}_{r',\infty,l} \to \mathfrak{X}_{r,\infty,l}$ can be normalised to give a morphism of sheaves on $\mathfrak{X}_{r,\infty,l}$*

$$\mathrm{tr}_\phi : \mathcal{O}_{\mathfrak{X}_{r',\infty,l}} \to S^{-3/p^{r-2}(p-1)}\mathcal{O}_{\mathfrak{X}_{r,\infty,l}} \subseteq \mathcal{O}_{\mathfrak{X}_{r,\infty,l}}[1/S].$$

In the limit, this gives rise to a continuous $\mathcal{O}_{\mathfrak{X}_{r,\infty,l}}$-linear section

$$\mathrm{tr} : \mathcal{O}_{\mathfrak{X}_{\infty,\infty,l}} \to S^{-3/p^{r-2}(p-1)}\mathcal{O}_{\mathfrak{X}_{r,\infty,l}} \subseteq \mathcal{O}_{\mathfrak{X}_{r,\infty,l}}[1/S].$$

Proof. This can be seen exactly like in [3], Corollaire 6.1.1 and 2. We repeat the proof in detail to make sure it also works on our larger weight space annuli with $l < 1$.

We first note that for both $r' < \infty$ and $r' = \infty$, it suffices to work locally on the base of the morphism $\mathfrak{X}_{r,\infty,l} \to \mathfrak{X}_{\mathbb{Z}_p}$. This is because if for any affine open $U_0 \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ with pullback U we have a trace $\mathrm{tr} : \mathcal{O}_{\mathfrak{X}_{r',\infty,l}}(U) \to S^{-\delta}\mathcal{O}_{r,\infty,l}(U) = A$ with $\delta = 3/p^r(p-1)$, then since the image of tr is bounded, we get the desired statement on any standard affine open $U(g \neq 0) \subseteq U$ for $g \in A$ by tensoring with $A\langle g^{-1}\rangle$ and completing S-adically. The

Lemma then holds on a basis of opens, and thus everywhere. It therefore suffices to prove the Lemma on any affine open $U_0 \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ on which ω is trivial. Let $A_{\mathbb{Z}_p} := \mathcal{O}_{\mathfrak{X}_{\mathbb{Z}_p}}(U_0)$.

We first verify part 1 for $r+2 \leq k < \infty$ and $r' = r+1$ using [3], Lemme 6.1. In the notation of said Lemma, we then take the algebra "S" to be $A_{\mathbb{Z}_p}$ which is formally smooth of dimension $d := 1$, and consider $\varphi : \mathbb{Z}_p[[T]] \to B := C_{k,l} = \mathbb{Z}_p[[T]][(1+T)^{1/p^k}]\langle p/S_k^{lp^k}\rangle$ via $T \mapsto S_k^{p^{k-m}}$ (we note that unless $k = m$, this is not a rescaling map). Then $C_{k,l}$ is integral and S_k-adically complete, thus $\varphi(T)$-adically complete. Since $l \in p^{-m}\mathbb{Z}_{>0}$ and thus $l \geq p^{-m}$, it also satisfies $p/\varphi(T) = p/S_k^{p^{k-m}} \in B$. We set $n := r - m - 1$. Then $n \geq 1$ since $r \geq m+2$. Moreover, $\varphi(T) = S_k^{p^{k-m}}$ admits an $n+1 = r-m$-th root, namely $\varphi(T)^{1/p^{n+1}} = S_k^{p^{k-r}}$ since $k \geq r+2$. Then $\varphi(T)^{1/p^n} = S_k^{p^{k-r+1}}$. Let $\pi := p/\varphi(T)^{1/p^n} = p/S_k^{p^{k-r+1}}$. Finally, let $f := \mathrm{Ha} \in A_{\mathbb{Z}_p}$ be any local lift of the Hasse invariant, $R := A_{\mathbb{Z}_p}\hat{\otimes}B$ and $R_n := R\langle X\rangle/(X\mathrm{Ha} - \varphi(T)^{1/p^n})$. With this setup, $\phi : A_{r,k,l} \to A_{r+1,k,l}$ is of the form

$$\phi : R_n = R\langle X\rangle/(X\mathrm{Ha} - \varphi(T)^{1/p^n}) \to R_{n+1} = R\langle X\rangle/(X\mathrm{Ha} - \varphi(T)^{1/p^{n+1}}).$$

Since $r+2 \leq k$, it reduces to the relative Frobenius mod $(\pi) = (p/S_k^{p^{k-r+1}}) \supseteq p/S_k^{p^{k-r}+p^{k-r-1}}$ by Proposition 6.4.1.2. We are thus allowed to apply [3], Lemme 6.1, which says that

$$\mathrm{tr}_\phi(A_{r+1,k,l}) \subseteq \varphi(T)^{-3/p^n} A_{r,k,l} = S_k^{-3p^{k-r+1}} A_{r,k,l}.$$

To extend to $k = \infty$, we note that for any $k \leq k' \in \mathbb{Z}_{\geq 0}$, the following diagram commutes

$$\begin{array}{ccc} A_{r+1,k,l} & \xrightarrow{\mathrm{Tr}_\phi} & A_{r,k,l} \\ \downarrow & & \downarrow \\ A_{r+1,k',l} & \xrightarrow{\mathrm{Tr}_\phi} & A_{r,k',l} \end{array}$$

by Proposition 6.4.1.3. The normalised traces for $k \to \infty$ are therefore compatible, and we may take their direct limit and complete S_k-adically to get a map $\mathrm{tr} : A_{r+1,\infty,l} \to S^{-3p^{-r+1}} A_{r,\infty,l}$. This is $A_{r,k',l}$-linear for all $k \leq k' < \infty$, in particular continuous, and thus $A_{r,\infty,l}$-linear. The same argument shows that tr restricts to the identity on $A_{r,\infty,l}$.

To prove the first part, we now iterate this trace: Since $\mathrm{tr}_{\phi_2\circ\phi_1} = \mathrm{tr}_{\phi_2} \circ \mathrm{tr}_{\phi_1}$, we have

$$\mathrm{tr}_{\phi_r^{r'}}(A_{r',\infty,l}) = \mathrm{tr}_{\phi_r} \circ \cdots \circ \mathrm{tr}_{\phi_{r'-1}}(A_{r',\infty,l}) \subseteq S^{-3p^{-r+1}} \cdots S^{-3p^{-r'+2}} A_{r,\infty,l}.$$

This factor equals $S^{-3\sum_{i=r}^{r'-1} p^{-i+1}}$ and using $\sum_{i=0}^{n} p^{-i} = (1-p^{-(n+1)})p/(p-1)$, we calculate

$$\sum_{i=r}^{r'-1} p^{-i+1} = p^{-r+1} \sum_{i=0}^{r'-r-1} p^{-i} = p^{-r+1}(1-p^{-(r'-r)})p/(p-1) < 1/p^{r-2}(p-1).$$

Since for any $a \leq b$ we have $(S^{-a}) \subseteq (S^{-b})$ by Notation 6.1.16, this shows that as desired,

$$\mathrm{tr}(A_{r',\infty,l}) \subseteq S^{-3/p^{r-2}(p-1)} A_{r,\infty,l}.$$

To deduce the second part, we note that the traces from the first part are continuous since they are $\mathcal{O}_{\mathfrak{X}_{r,\infty,l}}$-linear, and the codomain is bounded for the S-adic topology. They are moreover compatible for $r' \to \infty$. Since the codomain does not depend on r', we may again S_k-adically complete the direct limit over $r' \to \infty$ to obtain the desired section. □

Corollary 8.1.3. *Let $m \leq r+2 \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $l \leq l' \in 1/p^m\mathbb{Z}_{>0} \cup \{\infty\}$ and $I = [l, l']$. Let $\delta = 3/p^{r-2}(p-1)$. Then the trace $\mathrm{tr} : \mathcal{O}_{\mathfrak{X}_{\infty,\infty,l}} \to S^{-\delta}\mathcal{O}_{\mathfrak{X}_{r,\infty,l}}$ from Lemma 8.1.2 extends uniquely and functorially in I to a continuous $\mathcal{O}_{\mathfrak{X}_{r,\infty,I}}$-linear section of sheaves on $\mathfrak{X}_{r,\infty,I}$*

$$\mathrm{tr} : \mathcal{O}_{\mathfrak{X}_{\infty,\infty,I}} \to S^{-\delta}\mathcal{O}_{\mathfrak{X}_{r,\infty,I}}.$$

Proof. By the Cartesian diagram in Lemma 6.2.4.3.(ii), this follows from tensoring the map $\mathrm{tr}: \mathcal{O}_{\mathfrak{X}_{\infty,\infty,l}} \to S^{-\delta}\mathcal{O}_{\mathfrak{X}_{r,\infty,l}}$ from Lemma 8.1.2 over $C_{k,l}$ with $C_{k,I}$, then completing S-adically. This is functorial in I since base changing along $\mathfrak{W}_{k,I} \to \mathfrak{W}_{k,l}$ is. □

We can also deduce the result we needed for the U_p-operator on t-adic modular forms:

Corollary 8.1.4. *The maps $\phi^n : \mathfrak{X}'(p^{-n}\epsilon) \to \mathfrak{X}'(\epsilon)$ admit continuous $\mathcal{O}_K$-linear sections $\mathrm{tr}_{\phi^n} : \mathcal{O}_{\mathfrak{X}'(p^{-n}\epsilon)} \to t^{-3\epsilon p^3/(p-1)}\mathcal{O}_{\mathfrak{X}'(\epsilon)}$, compatible for varying n.*

Proof. This follows from Corollary 8.1.3 in the case $I = \{\infty\}$ and $r = 1$ by base change along the map $x : \mathrm{Spf}(\mathcal{O}_K) \to \mathfrak{W}_{\infty,\infty}$ which sends $S_3 \mapsto t^\epsilon$: Since $\mathfrak{X}_{2,\infty,\infty}$ is locally defined by an equation of the form $X\mathrm{Ha} - S_3$, the base change of $\mathfrak{X}_{2,\infty,\infty}$ along x is precisely of the form $b : \mathfrak{X}'(\epsilon) \to \mathfrak{X}_{2,\infty,\infty}$. Since x sends $(S) \mapsto (t^{\epsilon p^3})$, the base change of tr is of the form

$$\mathrm{tr}\,\hat{\otimes}_{\mathbb{F}_p[[T^{1/p^\infty}]]}\mathcal{O}_K : b_*\phi^n_*\mathcal{O}_{\mathfrak{X}'(p^{-n}\epsilon)} \to t^{-3\epsilon p^3/(p-1)}b_*\mathcal{O}_{\mathfrak{X}'(\epsilon)}.$$

Since the image is bounded, we may extend from a morphism of sheaves on $\mathfrak{X}_{r,\infty,\infty}$ to one of sheaves on $\mathfrak{X}'(\epsilon)$ like explained in the beginning of the proof of Proposition 8.1.2. □

Similarly, in the p-adic case, we now get the trace map promised in Lemma 2.2.6:

Corollary 8.1.5. *Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $1/2 > \epsilon \geq 0$ with $\epsilon \in \log|K|$. Then the map $q : \mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}(\epsilon)$ admits a continuous $\mathcal{O}_K$-linear section*

$$\mathrm{tr} : q_*\mathcal{O}_{\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a} \to p^{-3\epsilon p^3/(p-1)}\mathcal{O}_{\mathfrak{X}(\epsilon)}.$$

Proof. Let $r = m = l = 1$ and consider the point $x : \mathrm{Spf}(\mathcal{O}_K) \to \mathfrak{W}_\infty$ defined by sending $(1+T)^{1/p^{k+3}} \mapsto (1 + t^{\epsilon/p^k})^\sharp$ for any $k \geq 0$. This sends $(S_3) \mapsto ((1+t^\epsilon)^\sharp - 1) = (p^\epsilon)$ as one can calculate mod $p^{1-\epsilon}$ since $\epsilon < 1/2$. In particular, x already defines a point of $\mathfrak{W}_{\infty,1}$.

Since $\mathfrak{X}_{2,\infty,1}$ is locally defined by an equation of the form $X\mathrm{Ha} - S_3$, we see from the respective local descriptions that the base change of $\mathfrak{X}_{2,\infty,1}$ along x is $\mathfrak{X}(\epsilon)$. Similarly, $\mathfrak{X}_{\infty,\infty,1}$ after base change becomes $\mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$.

More generally, we see that x sends $(S^{p^{-k}}) = (S_k) \mapsto ((1 + t^{\epsilon/p^{k-3}})^\sharp - 1) = (p^{\epsilon/p^{k-3}})$, and sends $(S^\delta) \mapsto (p^{\delta\epsilon p^3})$ for any $\delta > 0$. In particular, x sends the factor $(S^{-3/p^{r-2}(p-1)}) = (S^{-3/(p-1)})$ to $(p^{-3\epsilon p^3/(p-1)})$. This shows that the trace map from Corollary 8.1.3 locally on $\mathfrak{X}_{2,\infty,1}$ base changes along x to the desired trace map. It can then be extended to a morphism of sheaves on $\mathfrak{X}(\epsilon)$ by the argument in the beginning of the proof of Proposition 8.1.2. □

8.2 Perfectoid rings and integrally sousperfectoid covers

Before we discuss sousperfectoidness, we recall the notion of perfectoid rings used in [53].

Definition 8.2.1 ([53], Definition 6.1.1)**.** A complete Tate ring R is perfectoid if R is uniform and there exists a pseudo-uniformisor $\varpi \in R$ such that $\varpi^p | p$ in R° and such that

$$\Phi : R^\circ/\varpi \to R^\circ/\varpi^p, \quad x \mapsto x^p$$

is an isomorphism. We note that it is equivalent to say that Φ is surjective:

Lemma 8.2.2. *Let A be a ring and let ϖ be a non-zero-divisor such that A is integrally closed in $A[1/\varpi]$. Then the Frobenius $A/\varpi \to A/\varpi^p$ is injective*

Proof. We can argue like in the proof of [48], Proposition 5.5: Given $x \in A$ with $x^p \in \varpi^p A$, the p-th power of $x/\varpi \in A[1/\varpi]$ is in A and thus so is x/ϖ. Thus $x \in \varpi A$. □

Definition 8.2.3. Let A be a ϖ-adically complete ring for some non-zerodivisor $\varpi \in A$. Then we say that A is integrally perfectoid if $A[1/\varpi]$ is perfectoid.

The following Lemma gives a technical variation of the Definition which we will need:

Lemma 8.2.4. *Let A be a ϖ-adically complete algebra for some non-zero-divisor $\varpi \in A$. Assume that A is integrally closed in $A[1/\varpi]$ and that there is $s \in A$ such that $s^p|p$, such that $\varpi|s$ and such that the Frobenius on A/s^p is surjective. Then A is integrally perfectoid.*

Proof. Since $R := A[1/\varpi]$ has an integrally closed ring of definition $R_0 := A$, it is uniform by Lemma A.2.1. The pseudo-uniformiser ϖ satisfies $\varpi^p|s^p|p$ like in Definition 8.2.1. It remains to see that $\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$ is surjective. Since by assumption we have surjections

$$R_0 \xrightarrow{F_{\mathrm{abs}}} R_0/s^p \twoheadrightarrow R_0/\varpi^p,$$

it is clear that the Frobenius $R_0/\varpi \to R_0/\varpi^p$ is surjective.

In order to pass from R_0 to R°, we first note that we may without loss of generality assume that ϖ has a p-th root $\varpi^{1/p}$ in R_0: Indeed, since Frobenius is surjective on R_0/ϖ^p, we can always find $v \in R_0$ such that $v^p = \varpi + \varpi^p t$ for some $t \in R_0$. We then have $\varpi' := v^p = \varpi(1 + \varpi^{p-1}t)$. Since R_0 is ϖ-adically complete, the factor $(1 + \varpi^{p-1}t)$ is a unit in R_0, and thus $(\varpi') = (\varpi)$. We may thus replace ϖ by ϖ' which admits the p-th root v.

We can now argue like in [7], Lemma 3.20: We first note that since $\varpi^p|p$, [7], Lemma 3.9 (iii)⇒(i) shows that the Frobenius on $R_0/\varpi p$ is surjective since it is surjective on R_0/ϖ^p.

To show that Φ is surjective, since $(\varpi^p) \supseteq (p)$, it suffices to prove that the Frobenius on R°/pR° is surjective. To see the latter, let now $x \in R^\circ$, then $\varpi x \in R_0$ by the assumption that R_0 is integrally closed. Since Frobenius is surjective on $R_0/\varpi p$, we can write $\varpi x = y^p + \varpi pz$ for some $y, z \in R_0$. Then $y' := y/\varpi^{1/p} \in R$ satisfies $y'^p = x - pz \in R^\circ$ and thus $y' \in R^\circ$. But then $x = y'^p + pz \in R^\circ$ shows that Frobenius is surjective on R°/pR°, as desired. □

Definition 8.2.5 ([53], Definition 6.3.1)**.** A complete Tate Huber ring (R, R^+) over $\mathbb{Z}_p$ is called sousperfectoid if there is a perfectoid ring $\widetilde{R}$ and a topological algebra homomorphism $R \hookrightarrow \widetilde{R}$ that has a splitting in topological R-modules. We call $R \hookrightarrow \widetilde{R}$ a sousperfectoid cover.

The reason for us to use sousperfectoidness is that it is a sheafiness criterion:

Proposition 8.2.6 ([53], §6.3.4)**.** *Sousperfectoid Huber rings are stably uniform, thus sheafy.*

Definition 8.2.7. Let $\varphi : A \to A_\infty$ be an injection of ϖ-torsionfree ϖ-adically complete rings for some $\varpi \in A$. We shall say that φ is an integrally sousperfectoid cover if $A_\infty[1/\varpi]$ is perfectoid and if there is an A-module homomorphism $s : A_\infty \to \varpi^{-n}A$ for some $n \in \mathbb{N}$ such that $s \circ \varphi = \mathrm{id}$. It is clear that $\varphi[1/\varpi] : A[1/\varpi] \to A_\infty[1/\varpi]$ is then a sousperfectoid cover. We shall call s a bounded generic section of φ.

We call a morphism of ϖ-adic formal schemes $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ an integrally sousperfectoid cover if it can be covered by morphisms $\mathrm{Spf}(A_\infty) \to \mathrm{Spf}(A)$ with $A \to A_\infty$ integrally perfectoid.

Corollary 8.2.8. *Let $\mathfrak{X}$ be a formal scheme that admits an integrally sousperfectoid cover. Then the adic generic fibre $\mathfrak{X}^{\mathrm{ad}}_\eta$ in the sense of [52], §2.2 is a sheafy adic space.*

8.3 Modular curves over perfected weight space are sousperfectoid

As the last step in this section, we now construct explicit integrally sousperfectoid covers of $\mathfrak{X}_{\infty,\infty,l}$ and use these to prove the main Theorem of this section:

Theorem 8.3.1. *Let $l \in \mathbb{Z}[1/p]_{>0}$ and let $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.*

1. $\mathcal{X}_{r,\infty,l} := (\mathfrak{X}_{r,\infty,l})^{\mathrm{ad}}_{\eta}$ *is a sousperfectoid adic space.*

2. *Let* $s : \mathcal{X}_{r,\infty,l} \to \mathfrak{X}_{r,\infty,l}$ *be the natural map, then we have* $s_* \mathcal{O}^+_{\mathcal{X}_{r,\infty,l}} = \mathcal{O}_{\mathfrak{X}_{r,\infty,l}}$.

Remark 8.3.2. One can extend this to $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, as well as to the Igusa curves $\mathfrak{IG}_{n,r,k,l}$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For $n = \infty$, this needs more arithmetic input, namely bounds on ramification in the Igusa tower, which is computed in [3], §3.2.2 for a different application. One can then use the relation of traces to the Dedekind different sheaves $\mathfrak{D}_{\mathfrak{IG}_{n,\infty,k,l}|\mathfrak{X}_{\infty,k,l}}$.

The starting point for the proof is that $\mathfrak{X}_{\infty,\infty,l}$ was defined as the inverse limit of maps which reduce to relative Frobenius mod p/S^l. The space $\mathfrak{X}_{\infty,\infty,l} \to \mathfrak{W}_{\infty,l}$ should therefore be "integrally relatively perfectoid", in the sense that $\mathfrak{X}_{\infty,\infty,l}$ becomes a formal model for a perfectoid space after passing to a suitable integrally perfectoid cover of $\mathfrak{W}_{\infty,l}$. We begin this section by discussing the explicit sousperfectoid cover of weight space we work with.

The perfectoid field we choose to base change to is $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$. An issue we need to take care of is that we cannot simply base change our formal models to the ring of integers $\mathbb{Z}_p\langle p^{1/p^\infty}\rangle$, but instead need to consider normalisations. Morally, we need to do this so that we can control the ring of bounded elements, i.e. the uniformity condition, although the actual place where this happens below will be disguised as an application of Lemma 8.2.4.

By a rescaling argument, it will be enough to consider the case of $l = 1$.

Definition 8.3.3. For any $d \leq k \in \mathbb{Z}_{\geq 0}$, let $C_{k,1,d} := \mathbb{Z}_p[p^{1/p^d}][[S_k]]\langle p^{1/p^d}/S_k^{p^{k-d}}\rangle$. This is clearly a $C_{k,1}$-algebra. We have natural maps of $C_{k,1}$-algebras $C_{k,1,d} \to C_{k+1,1,d}$ and also $C_{k,1,d} \to C_{k,1,d+1}$. We let $C_{\infty,1,d} := (\varinjlim_k C_{k,1,d})^\wedge$. We then set $C_{\infty,1,\infty} := (\varinjlim_d C_{\infty,1,d})^\wedge$.

To simplify notation, let us also write $\widetilde{C}_{\infty,1} := C_{\infty,1,\infty}$, we treat this as a cover of $C_{\infty,1}$.

Definition 8.3.4. For $d \leq k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let $\mathfrak{W}_{k,1,d} := \mathrm{Spf}(C_{k,1,d})$, $\widetilde{\mathfrak{W}}_{\infty,1} := \mathrm{Spf}(\widetilde{C}_{\infty,1})$.

We moreover set $\mathfrak{X}_{r,k,l,d} := \mathfrak{X}_{r,k,l} \times_{\mathfrak{W}_{k,l}} \mathfrak{W}_{k,l,d}$ and $\widetilde{\mathfrak{X}}_{r,\infty,l} = \mathfrak{X}_{r,\infty,l,\infty}$.

Lemma 8.3.5. *Let* $d \leq k < \infty$.

1. *The ring* $C_{k,1,d}$ *is normal.*

2. *For any* $r + 1 \leq k$, *the formal scheme* $\mathfrak{X}_{r,k,l,d}$ *is normal.*

Proof. We apply Serre's criterion to $C_{k,1,d} \cong C_{0,p^k,d} = \mathbb{Z}_p[p^{1/p^d}][[T]]\langle u\rangle/(uT^{p^{k-d}} - p^{1/p^d})$. Since $\mathbb{Z}_p[p^{1/p^d}][[T]]\langle u\rangle$ is a regular domain, $C_{k,1,d}$ is a complete intersection and satisfies S_2.

To verify R_1, let $\mathfrak{p} \subseteq C_{k,1,d}$ be a prime ideal of height 1.

If $T \notin \mathfrak{p}$, then $\mathfrak{p}$ corresponds by [21], Lemma 7.1.9, to a point of the smooth rigid space $\mathrm{Spf}(A)^{\mathrm{rig}}$ associated to the (p,T)-adic ring $A := \mathbb{Z}_p[p^{1/p^d}][[T]]$ by Berthelot's functor. But $\mathrm{Spf}(A)^{\mathrm{rig}}$ is the open unit disc over $\mathbb{Q}_p(p^{1/p^d})$, which is smooth, hence $A_\mathfrak{p}$ is regular.

If $T \in \mathfrak{p}$, then $A/T = \mathbb{F}_p[u]$ is a domain. Thus $\mathfrak{p} = (T)$ which shows that $A_\mathfrak{p}$ is regular.

For part 2, it suffices to see that for $\mathrm{Spf}(R) \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ the following ring is normal:

$$A := R\hat{\otimes}_{\mathbb{Z}_p} C_{k,l,d}\langle X\rangle/(X\mathrm{Ha} - S^{p^{-r-1}}) = R[p^{1/d}][[S_k]]\langle u, X\rangle/(uS_k^{p^{k-d}} - p^{1/d}, X\mathrm{Ha} - S_k^{p^{k-r-1}}).$$

For this we can argue like in part 1. The ring $R[p^{1/d}][[S_k]]\langle u, X\rangle$ is regular, and $uS_k^{p^{k-d}} - p^{1/d}, X\mathrm{Ha} - S_k^{p^{k-r-1}}$ is a regular sequence: This is because $R[p^{1/d}][[S_k]]\langle u\rangle/(uS_k^{p^{k-d}} - p^{1/d})$ is a domain, at least after assuming that R is. Thus A satisfies S_2.

To check R_1, let $\mathfrak{p}$ be a prime ideal of height 1. If $S_k \notin \mathfrak{p}$, we argue like in part 1, using that the rigid space $\mathcal{X}_{\mathbb{Q}_p(p^{1/p^d})} \times_{\mathbb{Q}_p(p^{1/p^d})} D$ is smooth, where $\mathcal{X}_{\mathbb{Q}_p(p^{1/p^d})}$ is the rigid modular curve over $\mathbb{Q}_p(p^{1/p^d})$ and D is the open unit disc over $\mathbb{Q}_p(p^{1/p^d})$ in the parameter S_k.

If $S_k \in \mathfrak{p}$, then $A/S_k = R/p[X,u]/(X\mathrm{Ha})$. Since R/p is a domain, we have $\mathfrak{p} = (S_k, X)$ or $\mathfrak{p} = (S_k, \mathrm{Ha})$. In the first case, we have $\mathfrak{p}A_\mathfrak{p} = S_k A_\mathfrak{p}$ since Ha becomes invertible in $A_\mathfrak{p}$. In the second, we have $\mathfrak{p}A_\mathfrak{p} = S_k A_\mathfrak{p}$ since X becomes invertible in $A_\mathfrak{p}$. Thus $A_\mathfrak{p}$ is regular. $\square$

Lemma 8.3.6. *Let $l < \infty$, $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Then $\mathcal{O}_{\widetilde{\mathfrak{X}}_{r,\infty,l}}$ is integrally closed in $\mathcal{O}_{\widetilde{\mathfrak{X}}_{r,\infty,l}}[1/S]$.*

Proof. This follows from Lemma 8.3.5 and Lemma A.2.2.1 in the limit $k, d, r \to \infty$. □

The following can be seen as an integral analogue of saying that $\mathcal{W}_{\infty,1}$ is preperfectoid in the sense of [37], §3.7.1.(a):

Lemma 8.3.7. *The ring $\widetilde{C}_{\infty,1}$ is integrally perfectoid, i.e. $\widetilde{C}_{\infty,1}[1/S]$ is perfectoid.*

Proof. We first recall that $C_\infty \xrightarrow{\sim} \mathbb{Z}_p[[T^{1/p^\infty}]]$ via $T \to S_\infty := \varinjlim S_m^{p^m}$. Since this morphism is $\mathbb{Z}_p$-linear, and by Lemma 6.1.15 the ideal generated by S_∞ in $C_{\infty,1}$ equals (S), this map extends for any $d \in \mathbb{Z}_{\geq 0}$ to an isomorphism

$$C_{\infty,1,d} = C_{\infty,1}[p^{1/p^d}]\langle p^{1/p^d}/S_d\rangle \xrightarrow{\sim} \mathbb{Z}_p[p^{1/p^d}][[T^{1/p^\infty}]]\langle (p/T)^{1/p^d}\rangle.$$

In the limit, this gives an isomorphism of topological algebras

$$\widetilde{C}_{\infty,1} = C_{\infty,1,\infty} \xrightarrow{\sim} \mathbb{Z}_p\langle p^{1/p^\infty}\rangle[[T^{1/p^\infty}]]\langle (p/T)^{1/p^\infty}\rangle.$$

The Frobenius on the latter is clearly surjective mod $p^{1/p}$ (this is [53], Example 6.1.5.4). □

Proposition 8.3.8. *The map $\widetilde{\mathfrak{W}}_{\infty,1} \to \mathfrak{W}_{\infty,1}$ is an integrally sousperfectoid cover. Consequently, $\mathcal{W}_{\infty,1} := (\mathfrak{W}_{\infty,1})^{\mathrm{ad}}_\eta$ is a sousperfectoid adic space.*

Proof. For any $n \in \mathbb{N}$, there is a normalised trace map $\mathrm{tr}_{\mathbb{Q}_p(p^{1/p^n})|\mathbb{Q}_p} : \mathbb{Z}_p[p^{1/p^n}] \to \mathbb{Z}_p$ given by sending $1 \mapsto 1$ and $p^{i/p^n} \to 0$ for $i = 1, \dots, p^n - 1$. In the completed limit, this defines a $\mathbb{Z}_p$-linear section of $\mathbb{Z}_p \to \mathbb{Z}_p\langle p^{1/p^\infty}\rangle$. In particular, this induces a $\mathbb{Z}_p$-linear section of

$$C_{\infty,1} \to C_{\infty,1}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p\langle p^{1/p^\infty}\rangle.$$

Next, observe that the natural inclusion map

$$C_{\infty,1}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p\langle p^{1/p^\infty}\rangle = C_\infty\langle p^{1/p^\infty}\rangle\langle p/S\rangle \hookrightarrow C_\infty\langle p^{1/p^\infty}\rangle\langle (p/S)^{1/p^\infty}\rangle = \widetilde{C}_{\infty,1}$$

admits a bounded generic section

$$\widetilde{C}_{\infty,1} \to S^{-1}C_{\infty,1}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p\langle p^{1/p^\infty}\rangle \tag{28}$$

since for any d, we have $p^{1/p^d}/S^{1/p^d} = (S^{-1}) \cdot S^{1-1/p^d} \cdot p^{1/p^d}$.

Since $\widetilde{C}_{\infty,1}$ is integrally perfectoid by Lemma 8.3.7, these two sections combine to show that the map $C_{\infty,1} \to \widetilde{C}_{\infty,1}$ is an integrally sousperfectoid cover, as desired. □

Lemma 8.3.9. *The relative Frobenius of $\widetilde{\mathfrak{X}}_{\infty,\infty,l}/(p/S^l) \to \widetilde{\mathfrak{W}}_{\infty,l}/(p/S^l)$ is an isomorphism.*

Proof. Let $\pi := p/S^l$. Let $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ be an affine open subspace on which ω is trivial. For any $1 \leq r, k \leq \infty$, write $A_{r,k,l}$ for $\mathcal{O}_{\mathfrak{X}_{r,k,l}}(U)$ like in Definition 6.2.1. Then by Proposition 6.4.1, for $k \geq r+2 \gg_l 0$, the transition map $\phi : A_{r+1,k,l} \to A_{r,k,l}$ reduces mod $\pi := p/S_k^{(p+1)p^{k-r-2}}$ to the Frobenius relative to $\mathfrak{W}_{k,l}$. The same is still true after the base change $C_{k,l} \to C_{k,l,d}$ since the base change of the relative Frobenius is the relative Frobenius. Let us denote the resulting morphism by $A_{r+1,k,l,d} \to A_{r,k,l,d}$. When we take the direct limit $k, r, d \to \infty$ for some fixed difference $\delta := k - r$, the resulting map $\varinjlim A_{r-1,k,l,d} \to \varinjlim A_{r,k,l,d}$ over $\varinjlim C_{k,l,d}$ still reduces to the relative Frobenius mod $\pi = p/S_k^{(p+1)p^{\delta-2}}$. On the other hand, it is clearly an isomorphism by index shifting.

By definition, we have $\mathcal{O}_{\widetilde{\mathfrak{X}}_{\infty,\infty,l}}(U) = (\varinjlim_{r,k,d} A_{r,k,l,d})^\wedge$. We are therefore left to see that the above morphism still reduces to relative Frobenius after completion. For this it suffices to check that we may commute reduction mod π and S-adic completion on $A_\infty := \varinjlim A_{r-1,k,l,d}$, since the completion of the relative Frobenius is clearly the relative Frobenius.

To see this, we write $\pi = (p/S_k^{lp^k}) \cdot S_k^b$ where $b = lp^k - (p+1)p^{k-r-2} \geq 0$ for $r \gg 0$. Since $p/S_k^{lp^k}$ is a non-zero-divisor mod S_k, this shows that for any $n \in \mathbb{N}$, the sequence

$$0 \to A_\infty/S_k^n \xrightarrow{\cdot\pi} A_\infty/S_k^{n+b} \to A_\infty/(\pi, S_k^{n+b}) \to 0$$

is exact. In the limit $n \to \infty$, this shows that S-adic completion commutes with reduction mod π on A_∞, as we wanted to see. □

Proof of Theorem 8.3.1. It suffices to treat the case of $l = 1$, the general case follows from rescaling and restriction. Our goal is to show that $\widetilde{\mathfrak{X}}_{\infty,\infty,1} \to \mathfrak{X}_{\infty,\infty,1}$ is an integrally sousperfectoid cover. In particular, the adic generic fibre $\mathcal{X}_{\infty,\infty,1}$ is then sousperfectoid, proving part 1 of the Theorem. Part 2 then follows from Lemma A.2.2.3.

By definition, $\widetilde{\mathfrak{X}}_{\infty,\infty,1} = \mathfrak{X}_{\infty,\infty,1} \times_{\mathfrak{W}_{\infty,1}} \widetilde{\mathfrak{W}}_{\infty,1}$. Since $\widetilde{\mathfrak{W}}_{\infty,1} \to \mathfrak{W}_{\infty,1}$ admits a bounded generic section, the same is true for $\widetilde{\mathfrak{X}}_{\infty,\infty,1} \to \mathfrak{X}_{\infty,\infty,1}$ by pullback. We are therefore left to see that the adic generic fibre of $\widetilde{\mathfrak{X}}_{\infty,\infty,1}$ is affinoid perfectoid locally on $\mathfrak{X}_{\infty,\infty,1}$.

To see this, let $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ be an affine open subspace on which ω is trivial. For any $1 \leq r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, write $A_{r,\infty,1} := \mathcal{O}_{\mathfrak{X}_{r,\infty,1}}(U)$. Let $\widetilde{A}_{\infty,\infty,1} := A_{\infty,\infty,1} \hat{\otimes}_{C_{\infty,1}} \widetilde{C}_{\infty,1}$. We want to show that this is integrally perfectoid. For this we wish to apply Lemma 8.2.4:

Let $(s) := p^{1/p}/S_2$ and $\varpi := S_2$. We check that these verify the conditions of the Lemma:

- We clearly have $(s^p) = (p/S_2^p) \supseteq (p)$ and thus $s^p|p$.
- To see that $\varpi|s$, recall that $(\varpi) = (S_2) = (S^{1/p^2})$. Then since $p^{1/p}/S^{1/p} \in \widetilde{A}_{\infty,\infty,1}$,
$$(s) = (p^{1/p}/S_2) = (p^{1/p}/S_2^p) \cdot (S_2^{p-1}) = (p^{1/p}/S^{1/p}) \cdot (\varpi^{p-1}) \subseteq (\varpi)$$
- The ϖ-adically complete $\widetilde{A}_{\infty,\infty,1} \subseteq \widetilde{A}_{\infty,\infty,1}[1/S]$ is integrally closed by Lemma 8.3.6.
- We are left to see that the absolute Frobenius on $\widetilde{A}_{\infty,\infty,1}/s^p$ is surjective: For this we first note that $\pi := p/S$ satisfies $(\pi) = (p/S) \subseteq (p/S^{1/p}) = (s^p)$. It therefore suffices to see that the absolute Frobenius on $\widetilde{A}_{\infty,\infty,1}/\pi$ is surjective. By Lemma 8.3.9, the Frobenius of $\widetilde{A}_{\infty,\infty,1}/\pi$ relative to $\widetilde{\mathfrak{W}}_{\infty,l}/\pi$ is surjective. By Lemma 8.3.7, $\widetilde{\mathfrak{W}}_{\infty,l}$ is integrally perfectoid and thus the absolute Frobenius on $\widetilde{\mathfrak{W}}_{\infty,l}/\pi$ is surjective. This combines to show that the absolute Frobenius on $\widetilde{A}_{\infty,\infty,1}/\pi$ is surjective, as desired.

Lemma 8.2.4 therefore applies to show that $\widetilde{A}_{\infty,\infty,1}[1/\varpi]$ is affinoid perfectoid. Thus $\widetilde{\mathfrak{X}}_{\infty,\infty,1}$ is integrally affinoid perfectoid locally on $\mathfrak{X}_{\infty,\infty,1}$. This finishes the proof of the Theorem. □

9 Modular forms over the perfected weight space

In this section, we build on the results of the last two sections and construct invertible modules of modular forms and perfectoid modular forms in families over the perfectified weight space. Like before, these come in three geometric flavours, namely line bundles $\mathfrak{w}_{r,k,l}$ on formal schemes of the form $\mathfrak{X}_{r,k,l}$, invertible $\mathcal{O}^+$-modules $\omega^+_{r,k,l}$ on $\mathcal{X}_{r,k,l}$ and invertible $\mathcal{O}$-modules $\omega_{r,k,l}$ on $\mathcal{X}_{r,k,l}$, and all have perfectoid relatives $\mathfrak{w}^{\mathrm{perf}}_{k,l}$, $\omega^{+,\mathrm{perf}}_{k,l}$ and $\omega^{\mathrm{perf}}_{k,l}$.

Also like before, we will show that one can recover $\mathfrak{w}$ from ω^+ and vice versa, but both versions of integral modular forms have their merits: The bundles $\mathfrak{w}$ come equipped with trace maps, and we do not know whether this extends to a morphism of the ω^+-sheaves on $\mathcal{X}_{r,k,l}$. On the other hand, we can glue instances of ω^+ over larger regions of weight space, for which it is not clear that this glueing is possible on formal models.

Recall that we parametrise annuli in the perfectified weight space in terms of subintervals of $[0,\infty]$ where 0 is the origin of the weight space disc, i.e. corresponds to the constant character $\kappa = 1$, and ∞ corresponds to the canonical boundary weight $\overline{\kappa}$. All bundles of modular forms will live over the region $(0, \infty]$, i.e. weight space punctured at the origin. In other words, we impose a condition $S \neq 0$ that allows us to work in a purely S-adic setting.

We begin this section by recalling the main results of [3] on these sheaves in the case of $k = 0$: In [3] Théorème 6.1 the authors construct for any $p \leq l \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $3 \leq r \in \mathbb{Z}_{\geq 0}$ line bundles of modular forms $\mathfrak{w}_{r,0,l}$ over what we denote by $\mathfrak{X}_{r,0,l}$. For $r' \geq r$ these satisfy $\mathfrak{w}_{r',0,l} = (\phi_r^{r'})^* \mathfrak{w}_{r,0,l}$. The construction goes via a sheaf of "perfected modular forms" $\mathfrak{w}_{0,l}^{\mathrm{perf}}$, which a posteriori can be described as $\mathfrak{w}_{0,l}^{\mathrm{perf}} = \phi^* \mathfrak{w}_{r,0,l}$ where $\phi : \mathfrak{X}_{\infty,0,l} \to \mathfrak{X}_{r,0,l}$ is the projection. In particular, the two are related by a trace map

$$\mathrm{tr} : \mathfrak{w}_{\infty,0,l} \to T^{-1} \mathfrak{w}_{r,0,l}.$$

The condition on l that $l \geq p$ is more restrictive than what we will need for our applications in the following. In order to extend to all $l \in [0, \infty]$, the authors construct in [3] Théorème 6.1 integral analytic sheaves over all of $\mathcal{W}$. By restriction, we then obtain an invertible $\mathcal{O}^+$-module $\omega_{r,0,l}^+$ on $\mathcal{X}_{r,0,l}$ for any $l \in \mathbb{Z}_{\geq 0}$. Here we need to slightly increase r to account for the fact that the bound on the Hasse invariant in [3],§3.3.1 is $|\mathrm{Ha}|^{p^{r+1}} \geq \max(|T|, |P|)$, whereas we exclusively work with $|\mathrm{Ha}|^{p^{r+1}} \geq |T|$. The reason the construction gives an invertible $\mathcal{O}^+$-module rather than a line bundle on the formal scheme $\mathfrak{X}_{r,0,l}$ is that the glueing happens on $\mathcal{X}_{r,0,l}$ rather than on $\mathfrak{X}_{r,0,l}$ (more precisely, on $\mathcal{W}_l$ rather than $\mathfrak{W}_l$).

On the other hand, with the preparations of the last section, it is easy to extend the definition of $\mathfrak{w}_{0,l}^{\mathrm{perf}}$ to any $0 < l$. The first goal of this section is now to compare this to the sheaves $\omega_{r,0,l}^+$ on $[l, \infty]$ also for $0 < l$, rather than for $p \leq l$, in a way extending the above constructions: In particular, we want the comparison to give rise to inclusion maps in the one direction, and trace maps in the other. This is achieved by Proposition 9.2.2 below.

Recall that the trace map does not preserve integrality: For a perfectoid modular form f, the trace $\mathrm{tr}\, f$ is in general only contained in $S^{-1}\mathfrak{w}$. In this section, we show that if a family $\mathfrak{f} \in \mathfrak{w}^{\mathrm{perf}}$ of perfectoid modular forms defined over the weight interval $[l, \infty]$ is such that the trace $\mathrm{tr}(\mathfrak{f})$ is integral at the boundary, then we can always ensure that $\mathrm{tr}(\mathfrak{f})$ is integral at the expense of slightly increasing l. This makes sense geometrically: If $\mathrm{tr}(\mathfrak{f})$ is bounded by 1 at ∞, it will be bounded by 1 on some neighbourhood of ∞. This neighbourhood turns out to be large, namely the region of weight space parametrised by $[l + \delta, \infty]$ for some small $\delta > 0$.

Once we can prove this, we have all the tools that we need to prove the main result of this section: Following a suggestion of Pilloni, we use the canonical lift of Proposition 7.3.4 to construct a "canonical lift" of t-adic modular forms, which extends any integral perfectoid modular form at the boundary into a canonical family of integral perfectoid modular forms over a large weight space annulus $[l, \infty]$ for $0 < l$. Using the result about integrality of the trace, we get an analogous statement for non-perfectoid modular forms. Finally, this allows us to construct the analogue of the $\sharp$-map of perfectoid algebras for modular forms.

9.1 Definition

We first define the sheaves of families of true p-and-t-adic modular forms in our setting. This can simply be done by pullback of the invertible modules defined by Andreatta–Iovita–Pilloni. For this we first need to introduce the modular curves over the entire weight space $\mathcal{W}$ as they are used in [3], since the condition on the Hasse invariant is slightly different. To tell them apart from our modular curves $\mathcal{X}$, we shall denote them by $\mathcal{M}$.

Definition 9.1.1. Throughout this section, we let $r_0 := 3$ if $p > 2$, or $r_0 := 5$ if $p = 2$.

Let $\mathfrak{M} := \mathfrak{W} \times_{\mathbb{Z}_p} \mathfrak{X}_{\mathbb{Z}_p}$ be the modular curve over $\mathfrak{W}$. Let $\mathcal{M}$ be its adic generic fibre over $\mathcal{W}$. For $r \geq r_0$, we set $\mathcal{M}_r := \mathcal{M}(|\mathrm{Ha}|^{p^{r+1}} \geq \max(|p|, |T|))$. Let $\mathcal{M}_{r,[0,1]} := \mathcal{M}_r(|p| \geq |T|)$ and let $\mathfrak{M}_{r,[0,1]}$ be the canonical formal model from [3], §3.3.2.

To be consistent with our notation of modular curves over weight space so far, we denote by $\omega^+_{r,0,0}$ the sheaf of integral modular forms on $\mathcal{M}_r$ from [3], Théorème 6.2. By construction, the sheaf $\omega^+_{r,0,0}$ has on the subspace $\mathcal{M}_{r,[0,1]}$ a formal model $\mathfrak{w}_{r,[0,1]}$ on $\mathfrak{M}_{r,[0,1]}$.

Lemma 9.1.2. *For any $m \in \mathbb{N}$, $l = p^{-m}$, $k \geq m$ and $r \geq m$, there are natural maps*

$$\begin{tikzcd} \mathcal{X}_{r,k,l} \arrow[r, dotted] \arrow[d] & \mathcal{M}_{r-m} \arrow[d] \\ \mathcal{W}_{k,l} \arrow[r] & \mathcal{W}_{0,0} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}_{r,k,[l,1]} \arrow[r, dotted] \arrow[d] & \mathcal{M}_{r-m,[0,1]} \arrow[d] \\ \mathcal{W}_{k,[l,1]} \arrow[r] & \mathcal{W}_{0,[0,1]} \end{tikzcd}$$

over $\mathfrak{X}_{\mathbb{Z}_p}$. The latter also has a formal model $\mathfrak{X}_{r,[l,1]} \to \mathfrak{M}_{r-m,[0,1]}$.

Proof. For the first map, recall that we have $\mathcal{M}_{r-m,[l,\infty]} = \mathcal{M}(|\mathrm{Ha}|^{p^{r-m+1}} \geq \max(|p|, |T|))$ and on the other hand $\mathcal{X}_{r,0,l} = \mathcal{M}(|\mathrm{Ha}|^{p^{r-m+1}} \geq |S_m|, |S_m| \geq |p|)$. Since we always have $|S_m| \geq |T|$, the conditions defining the latter are stronger, which gives the desired map (even an open immersion) for $k = 0$. For $k > 0$, the statement follows by base-change.

The second part follows from the first since we have $\mathcal{W}_k(|p| \geq |S_k^{p^k}|) \subseteq \mathcal{W}_k(|p| \geq |T|)$ by Lemma 6.1.17: We can therefore define the bottom map in the diagram as the composition $\mathcal{W}_{k,[l,1]} \subseteq \mathcal{W}_{k,[0,1]} \subseteq \mathcal{W}_k(|p| \geq |T|) \to \mathcal{W}_{0,[0,1]}$. This induces the top map by pullback.

Since $\mathfrak{X}_{r,0,l}$ is normal, the formal model can be defined locally using the $\mathcal{O}^+$-sheaf. $\square$

Definition 9.1.3. Let $r_0 \leq r \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$, $m \leq k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $l \in p^{-m}\mathbb{Z} \cup \{\infty\}$.

1. For $l \geq p$, we set $\mathfrak{w}_{r,k,l} := \varphi^* \mathfrak{w}_{r,0,p}$ where $\varphi : \mathfrak{X}_{r,k,l} \to \mathfrak{X}_{r,0,p}$ is the forgetful map.
2. Assume moreover that $r_0 + m \leq r$. Then we let $\omega^+_{r,k,l}$ be the pullback of $\omega^+_{r-m,0,0}$ along the map $\mathcal{X}_{r,k,l} \to \mathcal{M}_{r-m,[l,\infty]}$ from Lemma 9.1.2. Here for $k = \infty$, we use that the adic space $\mathcal{X}_{r,\infty,l}$ is still sheafy by Theorem 8.3.1.

   If $p \leq l$, this is the same as the analytication of $\mathfrak{w}_{r,k,l}$, i.e. the pullback along the map of locally ringed spaces $(\mathcal{X}_{r,k,l}, \mathcal{O}^+) \to \mathfrak{X}_{r,k,l}$. But the definition of $\omega^+_{r,k,l}$ is more general in the sense that it extends to $0 < l < p$.
3. The generic fibre $\omega_{r,k,l} := \omega^+_{r,k,l}[1/S]$ is the analytic sheaf of modular forms on $\mathcal{X}_{r,k,l}$.

It is clear from the definition that all these sheaves are again invertible modules.

With the preparations of the last section, there is a natural alternative definition of sheaves of modular forms for $r = \infty$, inspired by the definition of $\mathfrak{w}^{\mathrm{perf}}$ in characteristic p.

Definition 9.1.4. For any $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $l \in 1/p^k\mathbb{Z}_{>0} \cup \{\infty\}$, let

$$\mathfrak{w}^{\mathrm{perf}}_{k,l} := \mathcal{O}_{\mathfrak{IG}_{\infty,\infty,k,l}}[\kappa^{-1}], \quad \text{where } \kappa \text{ is the universal weight on } \mathfrak{W}_{k,l}.$$

For $k = \infty$, we denote its generic fibre on the adic space $\mathcal{X}_{\infty,\infty,l}$ by $\omega^{+,\mathrm{perf}}_{\infty,l}$.

Proposition 9.1.5 ([3], Proposition 6.4)**.** *The sheaf $\mathfrak{w}^{\mathrm{perf}}_{0,1} := \mathcal{O}_{\mathfrak{IG}_{\infty,\infty,0,1}}[\kappa^{-1}]$ is a line bundle on $\mathfrak{X}_{\infty,0,1}$. It is trivial on any affine open $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ on which ω is trivial.*

Proof. We only need to explain why the second part is true, because this is not part of the statement in [3], but it follows from its proof, which constructs an invertible section over any affine open of $\mathfrak{X}_{r,0,1}$ where the Hodge ideal is principal. Write $U = \mathrm{Spf}(R)$ and fix a trivialisation of ω on U. Then we can write $\mathrm{Ha} \in R$. On the pullback to $\mathfrak{X}_{r,0,1}$ for any $1 \leq r \in \mathbb{Z}_{\geq 0}$ we have $T \in (\mathrm{Ha})$. Since we are working over $l = 1$, and thus $p/T \in C_{k,1}$, this implies $p \in (\mathrm{Ha})$. But then the Hodge ideal $(p, \mathrm{Ha}) = (\mathrm{Ha})$ is indeed principal. $\square$

Our next goal is to relate for general $l \in \mathbb{Z}[1/p]_{>0}$ the analytic sheaf $\omega^+_{r,\infty,l}$ of modular forms on $\mathcal{X}_{r,\infty,l}$ to the formal sheaves $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ of perfectoid modular forms on $\mathfrak{X}_{\infty,\infty,l}$. In particular, we want to see that the analytification of $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ is simply the pullback of $\omega^+_{r,\infty,l}$ along $\mathcal{X}_{\infty,\infty,l} \to \mathcal{X}_{r,\infty,l}$. Until we know that the two are the same, however, we need to be able to tell them apart. We therefore introduce an alternative notation for the latter:

Definition 9.1.6. 1. Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $p \leq l \in p^{-k}\mathbb{Z}_{\geq 0} \cup \{\infty\}$. We emphasize the condition $p \leq l$. Set $\mathfrak{w}_{\infty,k,l} := \phi^*\mathfrak{w}_{r_0,k,l}$ where $\phi : \mathfrak{X}_{\infty,k,l} \to \mathfrak{X}_{r_0,k,l}$ is the natural map.

2. For $m \in \mathbb{Z}_{\geq 0}$, $l \in p^{-m}\mathbb{Z}_{>0}$ and $r := r_0 + m$, we let $\omega^+_{\infty,\infty,l} := \phi^*\omega^+_{r,\infty,l}$ where $\phi : \mathcal{X}_{\infty,\infty,l} \to \mathcal{X}_{r,\infty,l}$ is the projection, a morphism of adic spaces by Theorem 8.3.1.

The goal is now to show $\omega^{+,\mathrm{perf}}_{\infty,l} = \omega^+_{\infty,\infty,l}$. For $l \geq p$, this is clear by construction: The right hand side is defined by pullback of the sheaf $\mathfrak{w}_{r_0,0,1}$, which by [3], Théorème 6.4 is defined implicitly so that its pullback $\mathfrak{w}_{\infty,0,1}$ is equal to $\mathfrak{w}^{\mathrm{perf}}_{0,1}$. More generally:

Corollary 9.1.7. *For any $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $l \in 1/p^k\mathbb{Z}_{>0} \cup \{\infty\}$, $l \geq p$, we have*

$$\mathfrak{w}_{\infty,k,l} = \mathfrak{w}^{\mathrm{perf}}_{k,l}.$$

This is a line bundle, and is trivial on any affine open $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ on which ω is trivial.

Proof. It suffices to prove that $\mathfrak{w}^{\mathrm{perf}}_{k,l}$ is the pullback of $\mathfrak{w}^{\mathrm{perf}}_{0,p}$ along $\mathfrak{X}_{\infty,k,l} \to \mathfrak{X}_{\infty,0,p}$: With Proposition 9.1.5, this follows from Lemma 2.8.4 and the invariants statement that $(\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,k,l}})^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}_{\infty,k,l}}$ by Proposition 7.1.1. $\square$

It is not immediately clear that this is also true for $0 < l < p$. However, as we shall now discuss, this is essentially contained in the Théorème, and one can extend to the case of $k = \infty$ using the methods of [3], complemented by some tricks.

9.2 Extending towards the centre of weight space

The goal of this section is to prove the following comparison result about analytic bundles $\omega^+_{r,\infty,l}$ and the perfectoid formal models $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ over arbitrarily large annuli $[l,\infty]$ with $0 < l$:

For the statement, we update our setup to ensure that there is a sheaf $\omega^+_{r,\infty,l}$ on $\mathcal{X}_{r,\infty,l}$:

Assumption 9.2.1. Let $m \in \mathbb{Z}_{\geq 0}$ and $l \in p^{-m}\mathbb{Z}_{>0} \cup \{\infty\}$. Recall that we have defined $r_0 = 3$ if $p > 2$ and $r_0 = 5$ for $p = 2$. Let $r \in \mathbb{Z} \cup \{\infty\}$ with $r \geq m + r_0$. Set $\delta := 3/p^{r-2}(p-1)$.

Proposition 9.2.2. *Let m, l, r be as in Assumption 9.2.1. Let $s : \mathcal{X}_{r,\infty,l} \to \mathfrak{X}_{r,\infty,l}$, $s_\infty : \mathcal{X}_{\infty,\infty,l} \to \mathfrak{X}_{\infty,\infty,l}$ and $\phi : \mathfrak{X}_{\infty,\infty,l} \to \mathfrak{X}_{r,\infty,l}$ be the natural maps of locally ringed spaces.*

1. *There is a natural isomorphism of sheaves on $\mathcal{X}_{\infty,\infty,l}$*

$$\omega^+_{\infty,\infty,l} = \omega^{+,\mathrm{perf}}_{\infty,l}.$$

In particular, there is a natural isomorphism $s_{\infty*}\omega^+_{\infty,\infty,l} = \mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ of sheaves on $\mathfrak{X}_{\infty,\infty,l}$.

2. *There is an $\mathcal{O}_{\mathfrak{X}_{r,\infty,l}}$-linear map of sheaves on $\mathfrak{X}_{r,\infty,l}$*

$$\mathrm{tr} : \phi_*\mathfrak{w}^{\mathrm{perf}}_{\infty,l} \to S^{-\delta}s_*\omega^+_{r,\infty,l} \quad \textit{for } \delta := 3/p^{r-2}(p-1)$$

which is continuous, functorial in l and is the identity on $s_*\omega^+_{r,\infty,l} \subseteq s_*\omega^+_{\infty,\infty,l}$.

The reason we need to mix formal models and analytic sheaves in the above is that we do not know whether there is a line bundle "$\mathfrak{w}_{r,k,l}$" for $r < \infty$ and $0 < l < p$. More precisely, it is easy to define a sheaf, and by Lemma A.2.5 this will be identical to $s_*\omega^+_{r,k,l}$. But it is not clear that this is trivial locally on $\mathfrak{X}_{r,k,l}$ rather than locally on the much larger space $\mathcal{X}_{r,k,l}$. It is arguably one of the main results of [3] that $\mathfrak{w}_{r,0,l}$ is a line bundle for $l \geq p$.

Our first step for the proof is to show that $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ is a line bundle, since at this point we only know this for $l \geq 1$. While the proof for the case of $l \geq 1$ does not work verbatim for $l < 1$, the trick is to still work over $[1,\infty]$ but work with the sheaf of κ^{p^m}-invariants for any $m \in \mathbb{Z}$. After applying a rescaling isomorphism, this gives the same family of line bundles over the interval $[p^{-m},\infty]$, as desired.

The argument can be slightly extended to give for free the following pleasant observation: Recall that $\mathfrak{W}_\infty$ parametrises weights with a choice of arbitrary p^d-th roots. Then over $\mathfrak{X}_{\infty,\infty,l} \to \mathfrak{W}_\infty$, these give rise to "$p$-th roots" of the sheaves of perfectoid modular forms.

Corollary 9.2.3. *Let $l \in \mathbb{Z}[1/p]_{>0} \cup \{\infty\}$. Then for any $d \in \mathbb{Z}$, the sheaf*

$$(\mathfrak{w}^{\mathrm{perf}}_{\infty,l})^{\otimes 1/p^d} := \mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\infty,l}}[(\kappa^{1/p^d})^{-1}]$$

is a line bundle on $\mathfrak{X}_{\infty,\infty,l}$. It is trivial on any affine open $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ on which ω is trivial. In particular, $\mathfrak{w}^{\mathrm{perf}}_{\infty,l} := \mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\infty,l}}[\kappa^{-1}]$ is a line bundle with this property.

Proof. By Proposition 7.1.1, it suffices to show that $(\mathfrak{w}^{\mathrm{perf}}_{\infty,l})^{\otimes 1/p^d}$ has a non-vanishing section over U. It moreover suffices to consider the case of $d \geq 0$, the case of $d \leq 0$, follows from the case of $d = 0$ by raising the line bundle to the p^d-th tensor power.

To see the result for $d \geq 0$, let $m \in \mathbb{Z}_{\geq 0}$ be such that $l \in 1/p^m\mathbb{Z}_{\geq 0}$ and consider the rescaling isomorphism from Lemma 6.5.6

$$r_{m+d} : \mathfrak{IG}_{\infty,\infty,\infty,l} \xrightarrow{\sim} \mathfrak{IG}_{\infty,\infty,\infty,lp^{m+d}}, \quad S_{m+d} \leftarrow\!\shortmid T.$$

Let $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$ where ω is trivial. By Proposition 9.1.5, the bundle $\mathfrak{w}^{\mathrm{perf}}_{0,1}$ has a non-vanishing section f over U, and thus the same is true for $(\mathfrak{w}^{\mathrm{perf}}_{0,1})^{\otimes p^m}$ which has the non-vanishing section f^{p^m} over U. We can see f^{p^m} also as a section of $(\mathfrak{w}^{\mathrm{perf}}_{\infty,lp^{m+d}})^{\otimes p^m}$ by passing to $k \to \infty$ and passing from the weight space radius parameter $l = 1$ to $lp^{m+d} \geq 1$, namely by pullback along the transition map $\mathfrak{IG}_{\infty,\infty,\infty,lp^{m+d}} \to \mathfrak{IG}_{\infty,\infty,0,lp^{m+d}} \to \mathfrak{IG}_{\infty,\infty,0,1}$.

Since r_{m+d} is $\mathbb{Z}_p^\times$-linear and sends $1 + S_d = (1 + S_{m+d})^{p^m} \leftarrow\!\shortmid (1+T)^{p^m}$, we have for any $\gamma \in \mathbb{Z}_p^\times$, written as $\gamma = [\gamma]q^a$ for some $a \in \mathbb{Z}_p$ and $\gamma \in \mathbb{F}_p^\times$, that

$$r_{m+d}(\kappa(\gamma)^{p^m}) = r_{m+d}(\kappa([\gamma])(1+T)^{ap^m}) = \kappa([\gamma])(1+S_d)^a = \kappa([\gamma])\kappa^{1/p^d}(q)^a = \kappa^{1/p^d}(\gamma)$$

where we use that $\kappa([\gamma])^{p^m} = \kappa^{1/p^d}([\gamma])$ as $x^p = x$ in $\mathbb{F}_p^\times$. Consequently, $r_{m+d}(f^{p^m})$ satisfies

$$\gamma \cdot r_{m+d}(f^{p^m}) = r_{m+d}(\gamma \cdot f^{p^m}) = r_{m+d}(\kappa^{-1}(\gamma)^{p^m} f^{p^m}) = \kappa^{-1/p^d}(\gamma) r_{m+d}(f^{p^m}).$$

This shows that $r_{m+d}(f^{p^m})$ is an invertible section of $(\mathfrak{w}^{\mathrm{perf}}_{\infty,l})^{\otimes 1/p^d}$ over U as desired. □

proof of Proposition 9.2.2. By Theorem 8.3.1.2, the natural map $\mathcal{O}_{\mathfrak{X}_{\infty,\infty,l}} \to s_{\infty *}\mathcal{O}^+_{\mathcal{X}_{\infty,\infty,l}}$ of sheaves on $\mathfrak{X}_{\infty,\infty,l}$ is an isomorphism. Since by definition we have $\omega^{+,\mathrm{perf}}_{\infty,l} = s_\infty^*\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$, in order to to prove the first part it therefore suffices to show that there is a natural isomorphism

$$\omega^+_{\infty,\infty,l} = s_\infty^*\mathfrak{w}^{\mathrm{perf}}_{\infty,l}. \tag{29}$$

Since $\mathcal{X}_{\infty,\infty,l}$ is a sheafy adic space by Theorem 8.3.1.1, we may do so locally on $\mathcal{W}_{\infty,l}$, as long as we make sure that the construction glues. The same applies to the trace map in 2.

For the local construction of the desired maps, we need to consider subspaces of weight space parametrised by subintervals $I \subseteq [0,\infty]$ in the notation of §6.1. The interval we are working with in the Proposition is $[p^{-m}, \infty]$. Since the construction of ω_I^+ is functorial in I, it suffices to prove part 1 locally on intervals $I \subseteq [0, \infty]$ that cover $[p^{-m}, \infty]$: More precisely, to get equation (29), it suffices to show that we have a natural isomorphism

$$\omega^+_{\infty,\infty,I} = s_\infty^* \mathfrak{w}^{\mathrm{perf}}_{\infty,I}, \qquad \text{where} \qquad \mathfrak{w}^{\mathrm{perf}}_{\infty,I} := \mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\infty,I}}[\kappa^{-1}]$$

and where by "natural" we mean functorial with respect to the interval I. For the proof, we use that one can cover $[l, \infty]$ by intervals I for which $\omega^+_{r,\infty,I}$ has a canonical formal model $\mathfrak{w}_{r,\infty,I}$. It then suffices to prove that for each of these formal models we have

$$\mathfrak{w}^{\mathrm{perf}}_{\infty,I} = \phi^* \mathfrak{w}_{r,\infty,I}$$

because of the following calculation: For any such I we have a commutative diagram

$$\begin{tikzcd} \mathcal{X}_{\infty,\infty,I} \arrow[r, "\phi^{\mathrm{an}}"] \arrow[d, "s_\infty"] & \mathcal{X}_{r,\infty,I} \arrow[d, "s"] \\ \mathfrak{X}_{\infty,\infty,I} \arrow[r, "\phi"] & \mathfrak{X}_{r,\infty,I} \end{tikzcd}$$

and thus $s_\infty^* \mathfrak{w}^{\mathrm{perf}}_{\infty,I} = s_\infty^* \phi^* \mathfrak{w}_{r,\infty,I} = \phi^{\mathrm{an}*} s^* \mathfrak{w}_{r,\infty,I} = \phi^{\mathrm{an}*} \omega^+_{r,\infty,I} = \omega^+_{\infty,\infty,I}$, where we recall that the last step is precisely the definition of $\omega^+_{\infty,\infty,I}$. Since the maps s and ϕ commute with the transition maps in k, it suffices to prove that we have isomorphisms

$$\mathfrak{w}^{\mathrm{perf}}_{k,I} = \phi^* \mathfrak{w}_{r,k,I} \qquad \text{for some } k < \infty.$$

The cover of $[l, \infty]$ for which we have formal models as described is the following:

The case of $I = [p, \infty]$ and $k = 0$ is Théorème 6.4 of [3], which in our notation says that

$$\mathfrak{w}_{\infty,0,[p,\infty]} = \phi^* \mathfrak{w}_{r,0,[p,\infty]} \qquad \text{for } \phi : \mathfrak{X}_{\infty,0,p} \to \mathfrak{X}_{r,0,p}.$$

The case of $I = [1, p]$ and $k = 0$ is Proposition 6.6 in [3]: By the discussion in §6.5, [3], there is on $\mathfrak{X}_{r,0,[1,p]}$ a canonical formal model $\mathfrak{w}_{r,0,[1,p]}$ of $\omega^+_{r,0,[1,p]}$, and the Proposition says

$$\mathfrak{w}_{\infty,0,[1,p]} = \phi^* \mathfrak{w}_{r,0,[1,p]}.$$

To finish the proof of the first part, it thus suffices to consider the interval $[1/p^m, 1]$, for which by further subdivision it is sufficient to consider $I = [1/p^m, 1/p^{m-1}]$. Since we are free in our choice of k, we may for the moment assume $k = m$ and in particular $r \geq k + r_0$.

The interval I is contained in $[0, 1]$ for which Andreatta–Iovita–Pilloni construct in §5, [3] a sheaf $\mathfrak{w}_{[0,1]}$ on $\mathfrak{X}_{r,[0,1]}$ using the Pilloni-torsor $\mathfrak{F}_{n,r-m,0,[0,1]} \to \mathfrak{IG}_{n,r-m,0,[0,1]} \to \mathfrak{M}_{r-m,[0,1]}$. We would like to restrict this to $I \subseteq [0, 1]$, but in doing so we need to be careful: The construction for $\mathfrak{w}_{[0,1]}$ is slightly different in that it uses the p-adic topology on $\mathbb{Z}_p[[T]]\langle T/p\rangle$ rather than an S-adic topology, and a condition on the Hasse invariant in terms of p.

To deal with this, we observe that we have $\mathfrak{W}_{k,I} = \mathrm{Spf}(B_I)$ for

$$B_I = \mathbb{Z}_p[[T]][(1+T)^{1/p^k}]\langle S_m^{p^m}/p, p/S_m^{p^{m-1}}\rangle,$$

and in B_I we have $S_m^{p^m} \in (p)$ and $p \in (S_m^{p^{m-1}})$. Consequently, the S-adic and p-adic topology coincide, and $B_I[1/S] = B_I[1/p]$. Therefore $\mathfrak{X}_{r,k,I}$ is also S-adically complete, and we may equivalently regard $\mathfrak{IG}_{n,r,k,I}$ as being the normalisation in the generic fibre with respect to S as well as p. The rescaling isomorphism $r_k : C_k \xrightarrow{\sim} \mathbb{Z}_p[[T]]$ further induces isomorphisms

$$\begin{tikzcd} \mathfrak{IG}_{n,r,k,I} \arrow[r] \arrow[d, "\wr"'] & \mathfrak{X}_{r,k,I} \arrow[d, "\wr"'] \\ \mathfrak{IG}_{n,r-k,0,I'} \arrow[r] & \mathfrak{X}_{r-k,0,I'} \end{tikzcd}$$

where $I' := p^k I = [p^{k-m}, p^{k-m+1}] = [1,p]$. One can now define a Pilloni-torsor $\mathfrak{F}_{n,r,k,I} \to \mathfrak{IG}_{n,r,k,I}$ by the usual definition, and this fits into the above diagram by way of a rescaling isomorphism $\mathfrak{F}_{n,r,k,I} \to \mathfrak{F}_{n,r-k,0,I'}$ since rescaling identifies the Hodge ideals.

From now on, we can essentially follow the comparison of §6.5 [3] for the ideal $[1,p]$ and simply argue by rescaling: Andreatta–Iovita–Pilloni construct a map $\mathfrak{IG}_{\infty,\infty,0,[1,p]} \to \mathfrak{F}_{n,r-k,0,[1,p]}$ which they use to show that $\mathfrak{w}_{[1,p]}$ pulls back to $\mathfrak{w}^{\mathrm{perf}}_{0,[1,p]}$. This argument is essentially a relative version of the comparison we discussed in a perfectoid setting in §2.7.

Our constructions so far can be summarised in a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{F}_{n,r-m,k,[0,1]} \arrow[r] & \mathfrak{IG}_{n,r-m,k,[0,1]} \arrow[r] & \mathfrak{X}_{r-m,k,[0,1]} \\
\mathfrak{IG}_{\infty,\infty,k,I} \arrow[r, dotted] \arrow[d, "\wr"] & \mathfrak{F}_{n,r,k,I} \arrow[r] \arrow[u] \arrow[d, "\wr"] & \mathfrak{IG}_{n,r,k,I} \arrow[r] \arrow[u] \arrow[d, "\wr"] & \mathfrak{X}_{r,k,I} \arrow[u, "b"] \arrow[d, "r_k"', "\wr"] \\
\mathfrak{IG}_{\infty,\infty,0,[1,p]} \arrow[r] & \mathfrak{F}_{n,r-k,0,[1,p]} \arrow[r] & \mathfrak{IG}_{n,r-k,0,[1,p]} \arrow[r] & \mathfrak{X}_{r-k,0,[1,p]}
\end{tikzcd}$$

where the formal schemes in the top line are p-adic, and in the bottom and middle line are p-adic as well as S-adic. We may now define the dotted comparison arrow between the Igusa tower and the Pilloni-torsor as the rescaling of the one over $[1,p]$. The equivariance of the rescaling isomorphisms in particular implies that for $B = \mathbb{Z}_p^\times(1+p^n \mathrm{Hdg}_{r,k,I}^{-p^n/(p-1)})$ we have

$$(\mathcal{O}_{\mathfrak{F}_{n,r,k,I}})^B = \mathcal{O}_{\mathfrak{X}_{r,k,I}}.$$

We conclude that using the base change map b in the diagram we can describe $b^*\mathfrak{w}_{r,k,[0,1]}$ as being $\mathfrak{w}_{r,k,I} := \mathcal{O}_{\mathfrak{F}_{n,r,k,I}}[\kappa^{-1}] = b^*\mathfrak{w}_{r-m,k,[0,1]}$. The dotted arrow together with the equality $(\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,k,I}})^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}_{\infty,k,I}}$ deduced from the bottom row now induce the desired map

$$\mathfrak{w}^{\mathrm{perf}}_{k,I} = \phi^*\mathfrak{w}_{r,k,I}$$

as we wanted to see. This finishes the proof of part 1.

The second part of the Proposition now follows from the local construction of $\omega^{+}_{\infty,\infty,l}$: Any open $U \subseteq \mathfrak{X}_{r,\infty,l}$ after pullback to $\mathcal{X}_{r,\infty,l}$ can be covered by opens that come from pullback from $\mathfrak{X}_{r,\infty,I}$ for I of the form $I = I_1 = [p,\infty]$, $I = I_2 = [1,p]$ or $I = I_3 = [1/p^m, 1/p^{m-1}]$ for some m. It therefore suffices to prove that for each $I = I_i$ of this form, as well as on their intersections, there is a natural trace map

$$\mathrm{tr} : \mathfrak{w}^{\mathrm{perf}}_{\infty,I} \to S^{-\delta}\mathfrak{w}_{r,\infty,I},$$

functorial in I. But in each case we have a functorial isomorphism $\mathfrak{w}^{\mathrm{perf}}_{\infty,I} = \phi^*\mathfrak{w}_{r,\infty,I}$ as we have just seen, so this follows from functoriality of the trace map $\mathrm{tr} : \mathcal{O}_{\mathfrak{X}_{\infty,\infty,I}} \to S^{-\delta}\mathcal{O}_{\mathfrak{X}_{r,\infty,I}}$ from Corollary 8.1.3. This finishes the proof of part 2 $\square$

Finally in this section, we discuss the specialisation maps that associate to a family of modular forms over an interval of weight space a modular forms over a point of weight space: Corollary 9.2.3 lets us associate to any section of $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ a perfectoid modular forms at the boundary, as follows: Recall that we denote by $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t^{1/p^\infty}]]^\times$ the canonical weight.

Lemma 9.2.4. *Set $K = \mathbb{F}_p((T^{1/p^\infty}))$. Let $l \in \mathbb{Z}[1/p]_{>0}$, let $\epsilon = 1/p^{r+1}$, then there are natural closed immersions $i_r : \mathfrak{X}'(\epsilon) \hookrightarrow \mathfrak{X}_{r,\infty,l}$ and $i_\infty : \mathfrak{X}'(\epsilon)^{\mathrm{perf}} \hookrightarrow \mathfrak{X}_{\infty,\infty,l}$. These induce natural isomorphisms $i_r^*\mathfrak{w}_{r,\infty,l} = \mathfrak{w}^{\overline{\kappa}}$ and $i_\infty^*\mathfrak{w}^{\mathrm{perf}}_{\infty,l} = \mathfrak{w}^{\overline{\kappa},\mathrm{perf}}$ and a short exact sequence*

$$0 \to p/S^l \cdot \mathfrak{w}^{\mathrm{perf}}_{\infty,l} \to \mathfrak{w}^{\mathrm{perf}}_{\infty,l} \to i_*\mathfrak{w}^{\overline{\kappa},\mathrm{perf}} \to 0.$$

Proof. By the definitions we can identify $\mathfrak{X}_{r,\infty,\infty} = \mathfrak{X}'(\epsilon)$ and $\mathfrak{X}_{\infty,\infty,\infty} = \mathfrak{X}'(\epsilon)^{\mathrm{perf}}$. With these identifications, the first isomorphism is part of [3] Théorème 6.4. The second follows from pullback to the perfections. To deduce the exact sequence, we recall that by Lemma 6.5.5, the map $\mathfrak{X}_{\infty,\infty,\infty} \hookrightarrow \mathfrak{X}_{\infty,\infty,l}$ is a closed immersion cut out by the ideal (p/S^l) which defines $\mathfrak{W}_{\infty,\infty} \hookrightarrow \mathfrak{W}_{\infty,l}$. We therefore have a short exact sequence

$$0 \to p/S^l \cdot \mathcal{O}_{\mathfrak{X}_{\infty,\infty,l}} \to \mathcal{O}_{\mathfrak{X}_{\infty,\infty,l}} \to i_*\mathcal{O}_{\mathfrak{X}_{\infty,\infty,\infty}} \to 0.$$

Since $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ and $\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$ are locally trivial on $\mathfrak{X}_{\infty,\infty,l}$ by Corollary 9.2.3, the sequence displayed in the Lemma becomes locally isomorphic to the above exact sequence, and is thus exact. $\square$

Lemma 9.2.5. *Let m, l, r be as in Assumption 9.2.1. Let K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ be a family of non-trivial weights satisfying $\kappa_{n+1}^p = \kappa_n$. Assume $l > 0$ is small enough such that $(\kappa_n)_{n\in\mathbb{Z}_{\geq 0}}$ defines a point $x : \mathrm{Spf}(\mathcal{O}_K) \to \mathfrak{W}_{\infty,l}$. Let $\epsilon > 0$ be such that $|x(S)|^{1/p^{r+1}} = |p|^\epsilon$. Then base-change along x defines natural maps*

$$x_r : \mathfrak{X}(\epsilon) \to \mathfrak{X}_{r,\infty,l}, \qquad x_\infty : \mathfrak{X}_{\Gamma_0(p^\infty)}(\epsilon) \to \mathfrak{X}_{\infty,\infty,l}$$

and there are natural isomorphisms $\mathfrak{w}^\kappa = x_r^*\mathfrak{w}_{r,\infty,l}$ and $\mathfrak{w}^{\kappa,\mathrm{perf}} = x_\infty^*\mathfrak{w}^{\mathrm{perf}}_{\infty,l}$.

Proof. The first base change identification of the modular curves is clear from the local definitions. The second follows because $\phi : \mathfrak{X}_{r+1,\infty,l} \to \mathfrak{X}_{r,\infty,l}$ becomes identified with $\phi : \mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$, since both are generically defined by division by the canonical subgroup.

The isomorphism $\mathfrak{w}^\kappa = x_r^*\mathfrak{w}_{r,\infty,l}$ now follows from the construction in [3], §5 and our comparison Proposition 2.7.6. The second follows by pullback to infinite level. $\square$

9.3 Criteria for integrality of the trace

Lemma 9.3.1. *Let m, l, r be like in Assumption 9.2.1. Then inside $\mathcal{O}(\omega^{\mathrm{perf}}_{\infty,l})$, we have*

$$\mathcal{O}(\omega_{r,\infty,l}) \cap \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l}) = \mathcal{O}(\omega^+_{r,\infty,l}).$$

Proof. It suffices to prove this on some cover of $\mathcal{X}_{r,\infty,l}$. Since $\omega^+_{r,\infty,l}$ has a locally trivial formal model on $\mathfrak{X}_{r,k,J}$ for some $J \subseteq [l,\infty]$, it suffices to see that for affine open $U \subseteq \mathfrak{X}_{\mathbb{Z}_p}$,

$$\mathcal{O}(\mathfrak{X}_{r,\infty,J,U})[1/S] \cap \mathcal{O}(\mathfrak{X}_{\infty,\infty,J,U}) = \mathcal{O}(\mathfrak{X}_{r,\infty,J,U}).$$

But this follows from Lemma 6.5.7 for $n = 0$ and $r' = \infty$. $\square$

Proposition 9.3.2. *Let m, l, r be like in Assumption 9.2.1. Let $f \in \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l})$ be such that its image at the boundary, i.e. in $\mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,\infty})$, is already contained in $\mathcal{O}(\omega^+_{r,\infty,\infty})$. Then for any $l' \in 1/p^m\mathbb{Z}_{\geq 0}$ with $l' \geq l + \delta$ where $\delta = 3/p^{r-2}(p-1)$, the restriction of $\mathrm{tr}(f)$ to $\mathcal{X}_{r,\infty,l'} \hookrightarrow \mathcal{X}_{r,\infty,l}$ is integral, i.e. it is already contained in $\mathcal{O}(\omega^+_{r,\infty,l'}) \subseteq \mathcal{O}(\omega_{r,\infty,l'})$.*

Proof. Let $\Delta := l' - l.$, then $\Delta \geq \delta$ by the assumption on l' and thus $(S^{-\delta}) \subseteq (S^{-\Delta})$. By Proposition 9.2.2.1, we have $f \in \mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l})$. By Proposition 9.2.2.2, we then have $\mathrm{tr}(f) \in S^{-\delta}\mathcal{O}(\omega^+_{r,\infty,l}) \subseteq S^{-\Delta}\mathcal{O}(\omega^+_{r,\infty,l})$. Consider $g := f - \mathrm{tr}(f) \in S^{-\Delta}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l}) \subseteq \mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l})[1/S]$. Let $i : \mathfrak{X}_{\infty,\infty,\infty} \hookrightarrow \mathfrak{X}_{\infty,\infty,l}$ be the natural map. By Lemma 9.2.4, there is an exact sequence

$$0 \to p/S^l\mathfrak{w}^{\mathrm{perf}}_{\infty,l} \to \mathfrak{w}^{\mathrm{perf}}_{\infty,l} \to i_*\mathfrak{w}^{\mathrm{perf}}_{\infty,\infty} \to 0.$$

When we multiply this by $S^{-\Delta}$ and take global section, we obtain the left exact sequence

$$0 \to p/S^{l'}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l}) \to S^{-\Delta}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l}) \to S^{-\Delta}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,\infty}).$$

Denote by $\overline{f}$ the image of f in $\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,\infty})$. Then since the trace map is functorial in the weight space parameter, the image of $\mathrm{tr}(f)$ in $\mathcal{O}(\omega_{r,\infty,\infty})$ is equal to $\mathrm{tr}(\overline{f})$. But since $\overline{f} \in \mathcal{O}(\omega^{+}_{r,\infty,\infty})$ by assumption, we have $\mathrm{tr}(\overline{f}) = \overline{f}$. Consequently, the image of g in $S^{-\Delta}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,\infty})$ is $\mathrm{tr}(\overline{f}) - \overline{f} = \overline{f} - \overline{f} = 0$. By the second exact sequence, this implies that

$$g \in p/S^{l'}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l}).$$

Passing to l', the element $p/S^{l'}$ is contained in $\mathcal{O}(\mathfrak{W}_{\infty,l'}) \subseteq \mathcal{O}(\mathfrak{X}_{r,\infty,l'})$, and so in particular the image of g in $S^{-\Delta}\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l'})$ is already contained in $\mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l'})$. Thus $\mathrm{tr}(f) = f - g \in \mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l'}) = \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l'})$. Since $\mathrm{tr}(f) \in \mathcal{O}(\omega_{r,\infty,l'})$, we deduce from Lemma 9.3.1 that

$$\mathrm{tr}(f) \in \mathcal{O}(\omega_{r,\infty,l'}) \cap \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l'}) = \mathcal{O}(\omega^{+}_{r,\infty,l'}) \qquad \square$$

9.4 Canonical lifts at the boundary

We are now ready to prove the main result of this chapter: Every integral modular form at the boundary admits a canonical continuation into an integral family of modular forms over the annulus $\mathcal{W}_{\infty,l}$ in perfected weight space. This relies on the canonical lift of the perfect Igusa tower: In Proposition 7.3.4 we have constructed a canonical lift of $\mathfrak{IG}_{\infty,\infty,\infty,\infty}$ to a (p,T)-adic formal scheme $W\mathfrak{IG}_{\infty,\infty,\infty,\infty}$ equipped with a map $\mathfrak{IG}_{\infty,\infty,\infty,l} \to W\mathfrak{IG}_{\infty,\infty,\infty,\infty}$.

Theorem 9.4.1. *Let m, l, r be like in Assumption 9.2.1. Let $\delta := 3/p^{r-2}(p-1)$.*

1. *The specialisation map $\mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l}) \to \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,\infty})$ admits a multiplicative section*

$$[-] : \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,\infty}) \to \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l}).$$

2. *For $l \geq \delta + 1/p^{r-r_0}$, the specialisation map $\mathcal{O}(\omega^{+}_{r,\infty,l}) \to \mathcal{O}(\omega^{+}_{r,\infty,\infty})$ admits a canonical set-theoretic section*

$$[-]_r : \mathcal{O}(\omega^{+}_{r,\infty,\infty}) \to \mathcal{O}(\omega^{+}_{r,\infty,l}).$$

Definition 9.4.2. Let K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ and $\epsilon \in \mathbb{Q}$ be as in Lemma 9.2.5 with $|T_{\kappa_0}| \geq |p|$. Let $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t^{1/p^\infty}]]^\times$ be the canonical weight. Then composition with the maps from Lemma 9.2.5 defines natural maps

$$\sharp : M^{+,\mathrm{perf}}_{\overline{\kappa}}(\epsilon) \xrightarrow{[-]} \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,\infty}) \xrightarrow{\kappa} M^{+,\mathrm{perf}}_{\kappa}(\epsilon),$$
$$\natural : M^{+}_{\overline{\kappa}}(\epsilon) \xrightarrow{[-]_r} \mathcal{O}(\omega^{+}_{r,\infty,\infty}) \xrightarrow{\kappa} M^{+}_{\kappa}(\epsilon).$$

Remark 9.4.3. Using that the maps $\mathfrak{X}_{r,\infty,\infty} \leftarrow \mathfrak{X}_{\infty,\infty,\infty} \to W\mathfrak{X}_{\infty,\infty,\infty}$ identify the underlying topological spaces, the proof more generally shows that there are morphisms

$$[-] : \mathfrak{w}^{\mathrm{perf}}_{\infty,\infty} \to \varphi_*\mathfrak{w}^{\mathrm{perf}}_{\infty,l}, \qquad [-]_r : \phi_*\varphi^{-1}\mathfrak{w}_{r,\infty,\infty} \to \mathfrak{w}_{r,\infty,l}$$

of sheaves of sets, where $\varphi : \mathfrak{X}_{\infty,\infty,l} \to W\mathfrak{X}_{\infty,\infty,\infty}$ is the morphism from Proposition 7.3.4 applied in the case $n = 0$, and $\phi : \mathfrak{X}_{\infty,\infty,l} \to \mathfrak{X}_{r,\infty,l}$ is the natural projection.

Upon specialisation at a point $\kappa : \mathbb{Z}_p^\times \to \mathbb{Z}_p^{\mathrm{cyc}\times}$ with $\kappa \equiv \overline{\kappa}$ in $\mathbb{F}_p[[t^{1/p^\infty}]]/t = \mathbb{Z}_p^{\mathrm{cyc}}/p$, this simplifies to the pleasant statement that there is a morphism of sheaves of sets

$$\natural : \mathfrak{w}^{\overline{\kappa}} \to \mathfrak{w}^{\kappa},$$

because we can use the identifications $|\mathfrak{X}'(\epsilon)| = |\mathfrak{X}'(\epsilon)/t^{1-\epsilon}| = |\mathfrak{X}(\epsilon)/p^{1-\epsilon}| = |\mathfrak{X}(\epsilon)|$ to view both sides as sheaves on the same topological space.

Proof. 1. By Proposition 9.2.2.1, it suffices to produce a canonical section of the reduction

$$\mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l})[\kappa^{-1}] \to \mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty})[\overline{\kappa}^{-1}].$$

It is clear that we have a canonical $\mathbb{Z}_p^\times$-equivariant map

$$[-] : \mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,\infty}) \to \mathcal{O}(W\mathfrak{IG}_{\infty,\infty,\infty,\infty}).$$

which by Lemma 6.1.2 sends the weight $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[T^{1/p^\infty}]]^\times$ to the universal weight $\kappa = [\overline{\kappa}]$ on $\mathcal{W}_\infty$. By Proposition 7.3.4, we also have a $\mathbb{Z}_p^\times$-equivariant morphism

$$\mathcal{O}(W\mathfrak{IG}_{\infty,\infty,\infty,l}) \to \mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l}).$$

These compose to a multiplicative $\mathbb{Z}_p^\times$-equivariant map

$$\mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,\infty}) \xrightarrow{[-]} \mathcal{O}(W\mathfrak{IG}_{\infty,\infty,\infty,\infty}) \to \mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l})$$

which is a section of the reduction $\mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l}) \to \mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,\infty})$ by Proposition 7.3.4. By multiplicativity, $\mathbb{Z}_p^\times$-equivariance and $[\overline{\kappa}] = \kappa$, it restricts to

$$\mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l})[\overline{\kappa}^{-1}] \to \mathcal{O}(W\mathfrak{IG}_{\infty,\infty,\infty,l})[\kappa^{-1}] \to \mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l})[\kappa^{-1}].$$

2. Let $f \in \mathcal{O}(\omega^+_{r,\infty,\infty}) = \mathcal{O}(\omega^+_{r,\infty,\infty})$. By pulling back to $\mathcal{X}_{\infty,\infty,\infty} \to \mathcal{X}_{r,\infty,\infty}$, we may interpret f as a section of $\mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,\infty})$. By the first part of the Theorem with $m = r - r_0$ and $l_0 := 1/p^{r-r_0}$, there is then a canonical lift $\tilde{f} \in \mathcal{O}(\omega^{+,\mathrm{perf}}_{\infty,l_0}) = \mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l_0})$. Since $l \geq \delta + l_0$, Proposition 9.3.2 assures us that the restriction $\mathfrak{f}$ of the image $\mathrm{tr}_r(\tilde{f})$ under the trace map $\mathrm{tr}_r : \mathcal{O}(\mathfrak{w}^{\mathrm{perf}}_{\infty,l_0}) \to \mathcal{O}(\omega_{r,\infty,l_0})$ to $\mathcal{O}(\omega_{r,\infty,l})$ is already in $\mathcal{O}(\omega^+_{r,\infty,l})$. By Proposition 9.2.2.2, the trace is functorial in l, so the specialisation $\mathcal{O}(\omega^+_{r,\infty,l}) \to \mathcal{O}(\omega^+_{r,\infty,\infty})$ sends $\mathfrak{f}$ to $\mathrm{tr}_r(f) = f$. This shows that $\mathfrak{f}$ is a lift of f as desired. □

10 t-adic families of modular forms

Leaving the realm of p-adic families of modular forms behind, our next goal is to launch an analogous investigation into t-adic families of modular forms: In this section, we shall discuss how to interpolate the sheaves $\mathfrak{w}^\kappa$ for different weights of the form $\kappa : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ to get families of modular forms for equicharacteristic t-adic weight families.

Our main motivation to develop this theory is that we will use it in the proof of the Acyclicity Theorem. On the way, we also answer the question why t-adic families correspond to a single boundary point of the p-adic weight space: This is due to an additional "weight swap" symmetry that makes the t-adic setting substantially simpler than the p-adic one.

To make precise what we mean by t-adic families of weights, we first discuss weight space over $\mathbb{F}_p[[t]]$. Throughout we shall follow our earlier convention that characteristic p analogues of p-adic objects are denoted by the same symbol, adorned with an additional $'$.

In order to avoid confusion, we have also decided to denote the weight space parameter over $\mathbb{F}_p[[t]]$ by Z rather than T, although the latter plays precisely the same role over $\mathbb{Z}_p$.

10.1 The equicharacteristic weight space over $\mathbb{F}_p[[t]]$

Definition 10.1.1. Let $B = \mathbb{F}_p[[t]][[\mathbb{Z}_p^\times]]$ endowed with the profinite topology. As usual, the choice of the topological generator q of $\mathbb{Z}_p^\times/\mathbb{F}_p^\times$ defines an isomorphism of topological algebras $B \cong \mathbb{F}_p[[t]][(\mathbb{Z}/p\mathbb{Z})^\times]\hat{\otimes}_{\mathbb{F}_p[[t]]}\mathbb{F}_p[[t]][[Z]]$ where $\mathbb{F}_p[[t]][[Z]]$ is endowed with the (t, Z)-adic topology. The universal weight κ' over B is given by $q \mapsto 1 + Z$ plus a finite character. Let $\mathfrak{W}'_{\mathrm{full}} = \mathrm{Spf}(B)$. Let $\mathcal{W}'_{\mathrm{full}}$ be its t-adic generic fibre, this represents the functor

$$\{\text{adic spaces } \mathcal{Y} \text{ over } \mathbb{F}_p((t))\} \to \text{Sets}, \quad \mathcal{Y} \mapsto \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}(\mathcal{Y})^\times).$$

As in characteristic 0, we can see $\mathfrak{W}'_{\mathrm{full}}$ as a disjoint union of finitely may copies of $\mathfrak{W}' := \mathrm{Spf}(\mathbb{F}_p[[t]][[Z]])$ whose generic fibre is the open unit disc $\mathcal{W}'$ in the variable Z.

Like before, we work over affinoid open subspaces of $\mathcal{W}'$ parametrised by $l \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Our focus will this time be slightly different in that in the p-adic case we were focusing on boundary annuli of $\mathcal{W}'$, i.e. dealt with index intervals of the form $[l, \infty]$, whereas now we shall focus on the centre of $\mathcal{W}'$, i.e. work with the index interval $[0, l]$. We set:

$$\mathcal{W}'_{[0,l]} := \mathcal{W}'(|Z|^l \leq |t|) = \mathrm{Spa}(\mathbb{F}_p((t))\langle Z, Z^l/t\rangle),$$

$$\mathfrak{W}'_{[0,l]} := \mathrm{Spf}(\mathcal{O}^+(\mathcal{W}'_{[0,l]})) = \mathrm{Spf}(\mathbb{F}_p[[t]][[Z]]\langle Z^l/t\rangle).$$

There is of course a close algebraic analogy between $\mathbb{Z}_p[[T]]$ and $\mathbb{F}_p[[t]][[Z]]$. An important difference between the two is that in characteristic p there is a continuous automorphism exchanging the roles of t and Z, while there is no such thing over $\mathbb{Z}_p[[T]]$. More precisely:

Definition 10.1.2. Let $l \in \mathbb{Z}_{>0}$. We call weight swap map the morphism

$$s : \mathbb{F}_p[[t]] \to \mathbb{F}_p[[t, Z]]\langle Z^l/t\rangle, \quad t \mapsto Z.$$

Due to the presence of Z^l/t, this is continuous for the (t)-adic topology on the codomain.

Remark 10.1.3. We note that s sends the canonical weight $\overline{\kappa}$ at the boundary of $\mathcal{W}$ given by $q \mapsto 1+t$ to the universal weight κ' of $\mathcal{W}'$ given by $q \mapsto 1+Z$. We therefore think of s as being the bridge between the p-adic and the t-adic weight space. More precisely, the weight swap map will allow us to regard the boundary of $\mathcal{W}$ as being $\mathcal{W}'$ collapsed to a single point.

10.2 Modular curves over weight space in characteristic p

Like over $\mathcal{W}$, we define the t-adic modular curve $\mathcal{X}'_{[0,l]} := \mathcal{X}' \times_{\mathbb{F}_p((t))} \mathcal{W}'_{[0,l]}$ and for any $r \geq 0$ the open subspace $\mathcal{X}'_{r,[0,l]} = \mathcal{X}'_{[0,l]}(|Z| \leq |\mathrm{Ha}^{p^{r+1}}|)$. This has a canonical t-adic formal model $\mathfrak{X}'_{r,[0,l]}$, which over any affine open $\mathrm{Spec}(R) \subseteq X_{\mathbb{F}_p}$ where ω is trivial is given by

$$(\mathfrak{X}'_{r,[0,l]})_{|U} = \mathrm{Spf}(A) \quad \text{ where } A := R[[t, Z]]\langle Z^l/t, X\rangle/(X\mathrm{Ha}^{p^{r+1}} - Z)). \tag{30}$$

For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ (not necessarily satisfying $n \leq r$ in characteristic p) there is an Igusa scheme of level n over $\mathfrak{X}'_{r,[0,l]}$, which for our purposes we may just define as being

$$\mathfrak{IG}'_{n,r,[0,l]} := X_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \times_{X_{\mathbb{F}_p}} \mathfrak{X}'_{r,[0,l]} \to \mathfrak{X}'_{r,[0,l]},$$

motivated by Remark 6.3.7. Then $\mathfrak{IG}'_{\infty,r,[0,l]} = \varprojlim \mathfrak{IG}'_{n,r,[0,l]}$ inherits a natural $\mathbb{Z}_p^\times$-action.

The following Lemma compares modular curves over $\mathfrak{W}'$ to those over $\mathbb{F}_p[[t]]$ from §3.

Lemma 10.2.1. *There is a unique "weight swap" map of modular curves $\mathfrak{s} : \mathfrak{X}'_{r,[0,l]} \to \mathfrak{X}_r$ and a $\mathbb{Z}_p^\times$-equivariant map $\mathfrak{s}_\infty : \mathfrak{IG}'_{\infty,r,[0,l]} \to \mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}$ that fit into Cartesian squares*

$$\begin{tikzcd}
\mathfrak{IG}'_{\infty,r,[0,l]} \arrow[r] \arrow[d, "\mathfrak{s}_\infty"] & \mathfrak{X}'_{r,[0,l]} \arrow[r] \arrow[d, "\mathfrak{s}"] & \mathfrak{W}'_{[0,l]} \arrow[d, "s"] \\
\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)} \arrow[r] & \mathfrak{X}_r \arrow[r] & \mathrm{Spf}(\mathbb{F}_p[[t]]).
\end{tikzcd} \tag{31}$$

Proof. In the notation of (30), the map $\mathfrak{s}$ is locally given by the isomorphism

$$R[[t]]\langle X\rangle/(X\mathrm{Ha}^{p^{r+1}} - t)\hat{\otimes}_{\mathbb{F}_p[[t]],s}\mathbb{F}_p[[t, Z]]\langle Z^l/t\rangle \to R[[t, Z]]\langle Z^l/t, X\rangle/(X\mathrm{Ha}^{p^{r+1}} - Z),$$

induced by the map from $R[[t]]\langle X\rangle/(X\mathrm{Ha}^{p^{r+1}} - t)$ sending $t \mapsto Z$, $X \mapsto X$. For $\mathfrak{s}_\infty$, we first form the fibre product with $X_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \to X_{\mathbb{F}_p}$ and then the limit $n \to \infty$. $\square$

In order to follow our usual strategy from §2.8 to define line bundles of modular forms from cocycles, we need an invariance lemma. By the same issue as raised in Question 3.3.1, we are only able to obtain the following coarse version, which is "equation (16) in families":

Lemma 10.2.2. *Let $\mathfrak{IG}'_{\infty,r,[0,l]} \xrightarrow{q} \mathfrak{X}'_{r,[0,l]} \xrightarrow{\varphi} X_{\mathbb{F}_p}$ be the natural maps, then*

$$(\varphi_* q_* \mathcal{O}_{\mathfrak{IG}'_{\infty,r,[0,l]}})^{\mathbb{Z}_p^\times} = \varphi_* \mathcal{O}_{\mathfrak{X}'_{r,[0,l]}}.$$

Proof. We can see this by base change from the situation over $\mathbb{F}_p[[t]]$ discussed in §3.1: Recall that there we defined an Igusa formal scheme $\mathfrak{X}_{r,\mathrm{Ig}(p^n)}$ satisfying $\mathfrak{X}_{r,\mathrm{Ig}(p^n)} = X_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \times_{X_{\mathbb{F}_p}} \mathfrak{X}_r$. For any affine open $U = \mathrm{Spec}(R) \subseteq X_{\mathbb{F}_p}$ we have by Lemma 10.2.1:

$$\mathcal{O}_{\mathfrak{IG}'_{\infty,r,[0,l]}}(U) = \mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}(U) \hat{\otimes}_{\mathbb{F}_p[[t]],s} \mathbb{F}_p[[t,Z]]\langle Z^l/t\rangle$$

and $\mathcal{O}_{\mathfrak{X}'_{r,[0,l]}}(U) = \mathcal{O}_{\mathfrak{X}_r}(U) \hat{\otimes}_{\mathbb{F}_p[[t]],s} \mathbb{F}_p[[t,Z]]\langle Z^l/t\rangle$. By the discrete version of equation (16), $\mathcal{O}_{\mathfrak{X}_{r,\mathrm{Ig}(p^\infty)}}(U)^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}_r}(U)$. Lemma A.3.11.1 now allows us to commute $\hat{\otimes}$ and $(-)^{\mathbb{Z}_p^\times}$. □

10.3 Modular forms

We can now define the sheaf of t-adic families of modular forms. For the reasons laid out in Question 3.3.1, we need to follow the strategy of Definition 3.3.2.

Definition 10.3.1. Let $r_0 := 2$ if $p > 2$ and $r_0 := 3$ if $p = 2$. Let $l \in \mathbb{Z}_{\geq 0}$ and let $r \geq r_0$. Let $i : \mathfrak{X}'_{r,[0,l]} \to (X_{\mathbb{F}_p}, \varphi_* \mathcal{O}_{\mathfrak{X}'_{r,[0,l]}})$ be the natural map of ringed spaces. With notation like in Lemma 10.2.2, we first define

$$\mathfrak{w}'_{\mathrm{pre}} := \{f \in \varphi_* q_* \mathcal{O}_{\mathfrak{IG}'_{\infty,r,[0,l]}} \mid \gamma^* f = \kappa'^{-1}(\gamma) f \text{ for all } \gamma \in \mathbb{Z}_p^\times\}$$

where κ' is the universal weight. We then define a sheaf on $\mathfrak{X}'_{r,[0,l]}$ by setting $\mathfrak{w}'_{r,[0,l]} := i^* \mathfrak{w}'_{\mathrm{pre}}$.

We now prove that the entire theory of t-adic families of modular forms can be collapsed to and recovered from a single point: For this we use the weight swap from Lemma 10.2.1.

Proposition 10.3.2. *The map $\mathfrak{s}_\infty$ induces a canonical isomorphism of sheaves on $\mathfrak{X}'_{r,[0,l]}$*

$$\mathfrak{w}'_{r,[0,l]} = \mathfrak{s}^* \mathfrak{w}_r^{\overline{\kappa}}.$$

In particular, $\mathfrak{w}'_{r,[0,l]}$ is a line bundle.

Proof. The weight swap map $s : \mathfrak{W}'_{[0,l]} \to \mathrm{Spf}(\mathbb{F}_p[[t]])$ by definition sends the canonical character $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ to the universal character κ' on $\mathfrak{W}'$. The result thus follows from Lemma 2.8.4 applied to the pushforward to $X_{\mathbb{F}_p}$ of the left hand square of commutative diagram (31) above, where we use Lemma 10.2.2 for the necessary invariants statement: This first shows $\mathfrak{w}'_{\mathrm{pre}} = \mathfrak{s}^* \mathfrak{w}_{\mathrm{pre}}^{\overline{\kappa}}$, from which the displayed statement follows by pullback. □

10.4 General base rings

Let now K be any non-archimedean field extension of $\mathbb{F}_p((t))$. Then we can base change all of the above construction along $\mathrm{Spf}(\mathcal{O}_K) \to \mathrm{Spf}(\mathbb{F}_p[[t]])$ to obtain a weight space $\mathfrak{W}'_{[0,l],\mathcal{O}_K} = \mathrm{Spf}(\mathcal{O}_K[[Z]]\langle Z^l/t\rangle)$ over $\mathcal{O}_K$ and modular curves $\mathfrak{X}'_{r,[0,l],\mathcal{O}_K} \to \mathfrak{X}'_{[0,l],\mathcal{O}_K} \to \mathfrak{W}'_{[0,l],\mathcal{O}_K}$ with generic fibre $\mathcal{X}'_{r,[0,l],\mathcal{O}_K} \hookrightarrow \mathcal{X}'_{[0,l],\mathcal{O}_K}$, as well as a line bundle $\mathfrak{w}'_{[0,l],\mathcal{O}_K}$ on $\mathfrak{X}'_{r,[0,l],\mathcal{O}_K}$.

While carrying this out in detail, we also make a slight modification to the modular curve $\mathcal{X}'_{r,[0,l],\mathcal{O}_K}$ which we need for the Acyclicity Theorem: Recall that this space is defined by the condition $|Z| \leq |\mathrm{Ha}|^{p^r}$. We would like to have a version of this space in which the Hasse invariant is bounded *independently* of the weight variable Z. We therefore define:

Definition 10.4.1. Let $0 \leq \epsilon \in \log|K|$ be such that $\epsilon \cdot l \cdot p^{r+1} \leq 1$. Then we define

$$\mathcal{X}'^{\mathrm{u}}_{r,[0,l]}(\epsilon) := \mathcal{X}'_{[0,l],\mathcal{O}_K}(|t|^{\epsilon} \leq |\mathrm{Ha}|) \hookrightarrow \mathcal{X}'_{r,[0,l],\mathcal{O}_K}.$$

Here the "u" stands for "uniformly bounded", and the condition defines an open subspace since $|Z|^l \leq |t|$ and thus $|t|^{\epsilon} \leq |\mathrm{Ha}|$ implies the condition $|Z| \leq |Z|^{l\epsilon p^{r+1}} \leq |t|^{\epsilon p^{r+1}} \leq |\mathrm{Ha}|^{p^{r+1}}$ imposed on the right hand side. As usual, $\mathcal{X}'^{\mathrm{u}}_{r,[0,l]}(\epsilon)$ has a canonical formal model

$$h : \mathfrak{X}'^{\mathrm{u}}_{[0,l]}(\epsilon) \to \mathfrak{X}'_{r,[0,l]}$$

locally given by $\mathrm{Spf}(A)$ for $A := (R[[t,Z]]\hat{\otimes}_{\mathbb{F}_p[[t]]}\mathcal{O}_K)\langle Z^l/t, X\rangle/(X\mathrm{Ha} - t^{\epsilon})$.

We define the line bundle of modular forms on this by $\mathfrak{w}'^{\mathrm{u}}_{[0,l]} := h^*\mathfrak{w}'_{r,[0,l]}$.

We now wish to compare the above definition to the definition of modular forms we have made in §3. To do so, we restrict attention to the case of perfectoid K. This is not technically necessary, but it is the case we have considered there.

Proposition 10.4.2. *Let K be a perfectoid extension of $\mathbb{F}_p((t))$ and let $0 \leq \epsilon \in \log|K|$ with $\epsilon \leq 1/p^{r_0+1}$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight with $T_\kappa := \kappa(q) - 1$ such that $|T_\kappa| \leq |t|$. Then*

1. *The corresponding map $x_\kappa : \mathrm{Spf}(\mathcal{O}_K) \to \mathfrak{W}_{[0,1],\mathcal{O}_K}$ is a closed immersion defined by the principal ideal $(Z/t - T_\kappa/t)$. It induces a Cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}'(\epsilon) \arrow[r, hook, dotted, "i_\kappa"] \arrow[d] & \mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon) \arrow[d] \\ \mathrm{Spf}(\mathcal{O}_K) \arrow[r, hook, "x_\kappa"] & \mathfrak{W}_{[0,1],\mathcal{O}_K}. \end{tikzcd}$$

2. *The line bundle $\mathfrak{w}^\kappa$ on $\mathfrak{X}'(\epsilon)$ from Definition 3.3.6 is precisely $\mathfrak{w}^\kappa = i_\kappa^*\mathfrak{w}'^{\mathrm{u}}_{[0,1]}$.*

Note that we have implicitly set $l = 1$ in the Proposition. This is essential to ensure the the closed immersion x_κ is defined by a single element, which we will need later.

Proof. For part 1, it is clear that $\mathcal{O}_K\langle Z/t\rangle/(Z/t - T_\kappa/t) = \mathcal{O}_K$, where we use $|T_\kappa| \leq |t|$ to make sense of $T_\kappa/t \in \mathcal{O}_K$. To simplify notation, let us set $y_\kappa := Z/t - T_\kappa/t$.

To obtain the Cartesian diagram, we may work locally where we have an isomorphism

$$(R[[t,Z]]\hat{\otimes}_{\mathbb{F}_p[[t]]}\mathcal{O}_K)\langle Z/t, X\rangle/(X\mathrm{Ha} - t^{\epsilon}, y_\kappa) = (R[[t]]\hat{\otimes}_{\mathbb{F}_p[[t]]}\mathcal{O}_K)\langle X\rangle/(X\mathrm{Ha} - t^{\epsilon}).$$

For part 2, consider the composition $j : \mathfrak{X}'(\epsilon) \xhookrightarrow{i_\kappa} \mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon) \xrightarrow{h} \mathfrak{X}'_{r,[0,1]} \xrightarrow{\mathfrak{s}} \mathfrak{X}'_r$. Here the first two spaces are over $\mathcal{O}_K$ but the last two spaces are over $\mathbb{F}_p[[t]]$. This composition is just the natural base change map, plus the formal model of a restriction. It is therefore clear from the definition of $\mathfrak{w}^\kappa$ that we have $j^*\mathfrak{w}^{\overline{\kappa}} = \mathfrak{w}^\kappa$. By Proposition 10.3.2, we have $\mathfrak{s}^*\mathfrak{w}^{\overline{\kappa}} = \mathfrak{w}'_{r,[0,1]}$. We have $\mathfrak{w}'^{\mathrm{u}}_{[0,1]} = h^*\mathfrak{w}'_{[0,1]}$ by definition. This combines to show part 2 of the Lemma. $\square$

Corollary 10.4.3. *Assumptions as in Proposition 10.4.2, we have a short exact sequence*

$$0 \to \mathfrak{w}'^{\mathrm{u}}_{[0,1]} \xrightarrow{(Z/t - T_\kappa/t)\cdot} \mathfrak{w}'^{\mathrm{u}}_{[0,1]} \to i_{\kappa*}\mathfrak{w}^\kappa \to 0.$$

Proof. By Proposition 10.4.2.2, we have $i_\kappa^*\mathfrak{w}'^{\mathrm{u}}_{[0,1]} = \mathfrak{w}^\kappa$. This defines a map $\mathfrak{w}'^{\mathrm{u}}_{[0,1]} \to i_{\kappa*}\mathfrak{w}^\kappa$ by adjunction, and we can thus define the displayed sequence of maps. When restricted to any small enough open U on which $\mathfrak{w}'^{\mathrm{u}}_{[0,1]}$ is trivial, the sequence becomes isomorphic to

$$0 \to \mathcal{O}_{\mathfrak{X}'^{\mathrm{u}}_{[0,1]}|U} \xrightarrow{(Z/t - T_\kappa/t)\cdot} \mathcal{O}_{\mathfrak{X}'^{\mathrm{u}}_{[0,1]}|U} \to i_{\kappa*}\mathcal{O}_{\mathfrak{X}'(\epsilon)|U} \to 0.$$

By Proposition 10.4.2.1 this is precisely the restriction to U of the exact sequence associated to the closed immersion $i_\kappa : \mathfrak{X}'(\epsilon) \hookrightarrow \mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon)$. Thus also the above sequence is exact. $\square$

10.5 Canonical continuation into t-adic families

Another immediate consequence of Proposition 10.3.2 is that modular forms of canonical weight $\overline{\kappa}$ can be canonically continued into families over arbitrarily large weight space discs:

Corollary 10.5.1. *Let $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t]]^\times$ be the canonical weight and let $1 \le l \in \mathbb{Z}_{\ge 0}$. Then the isomorphism $\mathfrak{s}^*\mathfrak{w}_r^{\overline{\kappa}} = \mathfrak{w}'_{r,[0,l]}$ from Proposition 10.3.2 induces a canonical section of the specialisation map $\mathcal{O}(\mathfrak{w}'_{r,[0,l]}) \to \mathcal{O}(\mathfrak{w}_r^{\overline{\kappa}})$ associated to $\overline{\kappa}$.*

Proof. The composition $\mathfrak{X}_r \to \mathfrak{X}'_{r,[0,l]} \to \mathfrak{X}_r$ of the specialisation $i_{\overline{\kappa}}$ at $\overline{\kappa}$ with $\mathfrak{s}$ sends $T \mapsto Z \mapsto T$ and is therefore the identity. This induces morphisms

$$\mathcal{O}(\mathfrak{w}_r^{\overline{\kappa}}) \to \mathcal{O}(\mathfrak{s}^*\mathfrak{w}_r^{\overline{\kappa}}) \to \mathcal{O}(i_{\overline{\kappa}}^*\mathfrak{s}^*\mathfrak{w}_r^{\overline{\kappa}}) = \mathcal{O}(\mathfrak{w}_r^{\overline{\kappa}})$$

that compose to the identity. By Proposition 10.3.2, the second term equals $\mathcal{O}(\mathfrak{w}'_{r,[0,l]})$. $\square$

As a consequence, we can deduce the second main result of this section:

Proposition 10.5.2. *Let K be a perfectoid field extension of $\mathbb{F}_p((t))$. Let $\kappa_1, \kappa_2 : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be two non-trivial weights such that $\kappa_1 \equiv \kappa_2 \bmod t$ and $|T_{\kappa_1}| = |T_{\kappa_2}|$. Let $\epsilon = p^{-c}\epsilon_{\kappa_1}$ for some $c \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. Then inside $\mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon))/t$, we have an equality*

$$M^+_{\kappa_1}(\epsilon)/t = M^+_{\kappa_2}(\epsilon)/t$$

that is equivariant for the action of the integral Hecke-algebra on both sides.

Proof. We first note that $|T_{\kappa_1}| = |T_{\kappa_2}|$ implies $\epsilon_{\kappa_1} = \epsilon_{\kappa_2}$. In particular, the conditions on κ_1 and κ_2 are symmetric, and it therefore suffices by symmetry to construct a morphism $M^+_{\kappa_1}(\epsilon)/t \to M^+_{\kappa_2}(\epsilon)/t$ that commutes with the inclusions into $\mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon))/t$.

As usual, we can define $\iota : \mathbb{F}_p((t^{1/p^\infty})) \hookrightarrow K$ sending $t \mapsto T_{\kappa_1}$ such that $\kappa_1 = \iota \circ \overline{\kappa}$. By Proposition 4.3.3, and the conditions on ϵ, we have $M^+_{\kappa_1}(\epsilon) = M^+_{\overline{\kappa}}(p^{-r})\hat{\otimes}_\iota \mathcal{O}_K$ for $r = c + r_0$ and it therefore suffices to construct an ι-linear map

$$\varphi : M^+_{\overline{\kappa}}(p^{-r}) \to M^+_{\kappa_2}(\epsilon)/t.$$

The statement then follows from extending $\mathcal{O}_{K^\flat}/t$-linearly via ι.

Let $l \in \mathbb{Z}_{\ge 0}$ be large enough such that $|T_{\kappa_2}|^l \le |t|^2$. Then in particular, κ_2 corresponds to a point $x_2 : \mathrm{Spf}(\mathcal{O}_K) \to \mathfrak{W}'_{[0,l]}$. Since $|T_{\kappa_1}| = |T_{\kappa_2}|$, this induces a map $i_2 : \mathfrak{X}'(\epsilon) \to \mathfrak{X}'_{r,[0,l]}$ and pullback along i_2 defines a map $\mathcal{O}(\mathfrak{w}'_{r,[0,l]}) \to \mathcal{O}(\mathfrak{w}_r^{\kappa_2}) = M^+_{\kappa_2}(\epsilon)$. Combined with the canonical section s of Corollary 10.5.1, this gives a map of the desired form

$$M^+_{\overline{\kappa}}(p^{-r}) = \mathcal{O}(\mathfrak{w}_r^{\overline{\kappa}}) \xrightarrow{s} \mathcal{O}(\mathfrak{w}'_{r,[0,l]}) \to M^+_{\kappa_2}(\epsilon).$$

In order to see that this commutes with the maps to $\mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon))/t$, consider the following diagram, where we write x_{can} for the point of $\mathfrak{W}'_{[0,l]}$ defined by the canonical weight:

$$\begin{tikzcd} x_{\kappa_1} : \mathrm{Spf}(\mathcal{O}_K) \arrow[r, "\iota"] & \mathrm{Spf}(\mathbb{F}_p[[t]]) \arrow[r, "x_{\mathrm{can}}"] & \mathfrak{W}'_{[0,l]} = \mathrm{Spf}(\mathbb{F}_p[[t,Z]]\langle \frac{Z^l}{t} \rangle) \\ \mathrm{Spf}(\mathcal{O}_K/t) \arrow[u] \arrow[rr] & & \mathrm{Spf}(\mathcal{O}_K) \arrow[u, "x_{\kappa_2}"'] \end{tikzcd} \tag{32}$$

We claim that this diagram commutes: The top map sends $t \mapsto 0$, $Z \mapsto T_{\kappa_1}$ and $Z^l/t \mapsto T^l_{\kappa_1}/t = 0$, where the last equality uses that $|T^l_{\kappa_1}| \le |t|^2$ implies $T^l_{\kappa_1}/t \in t\mathcal{O}_K$. The bottom map similarly sends $t \mapsto 0$, $Z \mapsto T_{\kappa_2}$ and $Z^l/t \mapsto T^l_{\kappa_2}/t = 0$, where the last step uses that $|T_{\kappa_1}| = |T_{\kappa_2}|$. Since $\kappa_1 \equiv \kappa_2 \bmod t$ by assumption, this shows that the diagram commutes.

Consequently, for any $f \in M^+_{\underline{\kappa}}(p^{-r})$ with image $s(f) \in \mathcal{O}(\mathfrak{w}'_{r,[0,l]}) \subseteq \mathcal{O}(\mathfrak{IG}'_{\infty,r,[0,l]})$, the specialisation at κ_1, which coincides with the image of f in $M^+_{\kappa_1}(\epsilon)$, agrees with the specialisation at κ_2 after reducing mod t. This gives the desired morphism $\varphi : M^+_{\kappa_1}(\epsilon)/t \hookrightarrow M^+_{\kappa_2}(\epsilon)/t$ inside $\mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon))/t$.

To see that φ is Hecke-equivariant, we first note that via the weight swap map we can immediately define Hecke operators on t-adic families of modular forms, i.e. on $\mathcal{O}(\mathfrak{w}'_{r,[0,l]})$, by base change. With this definition of Hecke operators on families, the specialisation maps are clearly Hecke-equivariant because weight-swap followed by specialisation amounts to a base change of the form $\mathbb{F}_p((t)) \hookrightarrow K$, and base-change is indeed Hecke-equivariant.

Since by construction of φ, for any $f \in M^+_{\kappa_1}(\epsilon)$ both f and $\varphi(f) \in M^+_{\kappa_2}(\epsilon)$ are specialisations of the same family of t-adic modular forms along the maps in diagram (32), this shows that φ is equivariant for the action of the integral Hecke algebra, as desired. $\square$

11 Almost acyclicity of perfectoid modular sheaves

The goal of this section is to prove the following Almost Acyclicity Theorem:

Definition 11.0.1. Let R be a ring with an almost setting, i.e. an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$. Then a sheaf F of R-modules on a topological space X is called almost acyclic if

$$H^q(X, F) \overset{a}{=} 0 \text{ for all } q > 0.$$

For example, for any affinoid perfectoid space X, the integral structure sheaf $\mathcal{O}^+_X$ on X is almost acyclic by [48], Theorem 6.3.(iv).

Theorem 11.0.2. *Let K be a perfectoid field. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight and $0 \leq \epsilon \leq \epsilon_\kappa$.*

1. *If K is of characteristic p, then the sheaf $\omega^{\kappa,+,\mathrm{perf}}$ on $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$ is almost acyclic.*
2. *If K is of characteristic 0, then the sheaf $\omega^{\kappa,+,\mathrm{perf}}$ on $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ is almost acyclic.*

Note that for the constant weight $\kappa = 1$, these statements specialise to the almost acyclicity of the structure sheaf $\mathcal{O}^+$, because $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$ and $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ are affinoid perfectoid.

Our proof of the Theorem follows the general strategy of the proof of Scholze's acyclicity theorem: In order to explain our method of proof, let us briefly recall the strategy in [48]: One reduces to a computation of Čech cohomology for an open cover $\mathfrak{U}$, where one reduces to the case of characteristic p, and then to the case that X is the perfection $X = X_0^{\mathrm{perf}}$ of a reduced affinoid rigid space X_0. We can then write the Čech complex $\check{C}^*(\mathfrak{U}, \mathcal{O}^+_X)$ as the completed ind-perfection of the Čech complex $\check{C}^*(\mathfrak{U}, \mathcal{O}^+_{X_0})$. Since for an $\mathbb{F}_p[[t]]$-module M which is annihilated by some t^n, the ind-perfection of M is almost zero as an $\mathbb{F}_p[[t^{1/p^\infty}]]$-module, it now suffices to see that $\check{H}^*(\mathfrak{U}, \mathcal{O}^+_{X_0})$ is annihilated by some t^n. But this follows from a topological algebra lemma, which we record as Lemma A.3.2, and which is essentially a combination of the Open Mapping Theorem with Tate's Acyclicity Theorem.

In order to adapt this strategy to our setting, the basic idea is to replace $\mathcal{O}^+_{X_0}$ by $\omega^{\kappa,+}$. We can then write $\omega^{\kappa,+,\mathrm{perf}}$ as the ind-perfection of $\omega^{\kappa,+}$, in the following sense: Using that $F^*\omega^{\kappa,+} = \omega^{\kappa^p,+}$, where F denotes the absolute Frobenius on $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$ we can write $\omega^{\kappa,+,\mathrm{perf}}$ as the t-adically completed limit of the directed system

$$\omega^{\kappa,+} \xrightarrow{F} \omega^{\kappa^p,+} \xrightarrow{F} \omega^{\kappa^{p^2},+} \xrightarrow{F} \dots.$$

The problem now reduces to showing that we can find a cover of $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$ for which the Čech cohomology of the $\omega^{\kappa^{p^n},+}$ is killed by some t^N uniformly for all n. This is where we draw on the theory of t-adic families of modular forms developed in the last section.

11.1 Perfecting modular forms and acyclicity in characteristic p

We now make precise the strategy laid out above, starting with the perfection of $\mathfrak{w}^{\kappa}$.

Lemma 11.1.1. *Let K be a perfectoid field of characteristic p and let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight. Then the sheaf $\mathfrak{w}^{\kappa,\mathrm{perf}}$ on $\mathfrak{X}'(\epsilon)^{\mathrm{perf}}$ can be regarded as the completed ind-perfection of $\mathfrak{w}^{\kappa}$, in the sense that there is a natural isomorphism of sheaves on $|\mathfrak{X}'(\epsilon)^{\mathrm{perf}}| = |\mathfrak{X}'(\epsilon)|$*

$$\mathfrak{w}^{\kappa,\mathrm{perf}} = (\varinjlim_{F_{\mathrm{abs}}} \mathfrak{w}^{\kappa^{p^n}})^{\wedge}.$$

Proof. By Lemma 3.6.4, we have a natural isomorphism $F_{\mathrm{abs}}^{*}\mathfrak{w}^{\kappa} = \mathfrak{w}^{\kappa^p}$, which by adjunction induces $F_{\mathrm{abs}} : \mathfrak{w}^{\kappa} \to F_{\mathrm{abs}*}\mathfrak{w}^{\kappa^p} = \mathfrak{w}^{\kappa^p}$. We now iterate this map in the tower

$$\mathfrak{X}(\epsilon)^{\mathrm{perf}} \to \cdots \to \mathfrak{X}'(\epsilon) \xrightarrow{F_{\mathrm{abs}}} \mathfrak{X}'(\epsilon).$$

We know that $\mathfrak{w}^{\kappa,\mathrm{perf}}$ is the pullback to the limit. By adjunction, we get a natural map

$$\varinjlim_{F_{\mathrm{abs}}} \mathfrak{w}^{\kappa^{p^n}} \subseteq \mathfrak{w}^{\kappa,\mathrm{perf}}.$$

Since this becomes an equality upon completion locally on any affine open subspace where these sheaves are trivial, the same is true globally. □

The following key lemma is our main application of t-adic families of modular forms:

Lemma 11.1.2. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be any weight. Let $0 \le \epsilon \le \epsilon_\kappa$. Then there is $N \in \mathbb{Z}_{\ge 0}$ such that $H^q(\mathfrak{X}'(\epsilon), \mathfrak{w}^{\kappa^{p^n}})$ is annihilated by t^N for all $n \in \mathbb{Z}_{\ge 0}$ and all $q > 0$.*

Proof. For typesetting convenience, let us write $\kappa_n := \kappa^{p^n}$ and $\mathfrak{w}_n := \mathfrak{w}^{\kappa_n}$. The idea for the proof is that the line bundle $\mathfrak{w}'^{\mathrm{u}}_{[0,1]}$ on $\mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon)$ from Definition 10.4.1 "sees" all but finitely many of the $\mathfrak{w}_n$ and is therefore able to bound the torsion in their cohomology uniformly.

Since $\mathfrak{w}_n$ is a line bundle on the formal scheme $\mathfrak{X}'(\epsilon)$ whose generic fibre is a reduced affinoid rigid space, Lemma A.3.2 says that for any n there is $N \in \mathbb{N}$ such that all $H^q(\mathfrak{X}'(\epsilon), \mathfrak{w}_n)$ for $q > 0$ are annihilated by t^N. We need to show that N can be chosen uniformly.

Recall that for growing n, the weights κ_n converge to the centre of weight space $\mathcal{W}'$. For $n \gg 0$, the κ_n therefore correspond to points of $\mathcal{W}'_{[0,1]}$. Since we may discard finitely many $\mathfrak{w}_n$, we may assume without loss of generality that all κ_n correspond to points of $\mathcal{W}'_{[0,1]}$. By Lemma 10.4.2, they then correspond to closed immersions $x_n : \mathrm{Spf}(\mathbb{F}_p[[t]]) \to \mathfrak{W}'_{[0,1]}$ cut out by the principal ideal generated by $f_n := Z/t - T_n/t$ where $T_n := \kappa_n(q) - 1$.

Since the generic fibre $\mathcal{X}'^{\mathrm{u}}_{[0,1]}(\epsilon)$ is also a reduced affinoid rigid space, Lemma A.3.2 shows that there is $m > 0$ for which all $H^q(\mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon), \mathfrak{w}'^{\mathrm{u}}_{[0,1]})$ for $q > 0$ are annihilated by t^m.

In order to bound the t-torsion in $H^q(\mathfrak{X}'(\epsilon), \mathfrak{w}_n)$ by the t-torsion in $H^q(\mathfrak{X}'^{\mathrm{u}}_{[0,l]}(\epsilon), \mathfrak{w}'^{\mathrm{u}}_{[0,1]})$, recall from Corollary 10.4.3 that we have a short exact sequence

$$0 \to \mathfrak{w}'^{\mathrm{u}}_{[0,1]} \xrightarrow{f_n} \mathfrak{w}'^{\mathrm{u}}_{[0,1]} \to i_{n*}\mathfrak{w}_n \to 0.$$

Since i_n is a closed immersion, it follows for example from the Leray spectral sequence that

$$H^q(\mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon), i_{n*}\mathfrak{w}_n) = H^q(\mathfrak{X}'(\epsilon), \mathfrak{w}_n).$$

Combining this with the short exact sequence above, we see from the corresponding long exact sequence of cohomology on $\mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon)$ that for any $q > 0$ we have a short exact sequence

$$0 \to H^q(\mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon), \mathfrak{w}'^{\mathrm{u}}_{[0,1]})/f_n \to H^q(\mathfrak{X}'(\epsilon), \mathfrak{w}_n) \to H^{q+1}(\mathfrak{X}'^{\mathrm{u}}_{[0,1]}(\epsilon), \mathfrak{w}'^{\mathrm{u}}_{[0,1]})[f_n] \to 0.$$

In particular, since the two outer terms are t^m-torsion, the term in the middle is t^{2m}-torsion. Setting $N = 2m$, this proves the Lemma. □

proof of Theorem 11.0.2.1. We can now prove that $\omega^{\kappa,+,\mathrm{perf}}$ is almost acyclic. Let $\mathfrak{U}$ be a cover of $X_{\mathbb{F}_p}$ by affine opens with affine intersection on which ω becomes trivial. Then also the invertible $\mathcal{O}^+$-module $\omega^{\kappa,+,\mathrm{perf}}$ is trivial on the pullbacks of these open subspaces to $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$. Since the integral structure sheaf $\mathcal{O}^+$ of $\mathcal{X}'(\epsilon)^{\mathrm{perf}}$ is almost acyclic, the same is true for the restriction of $\omega^{\kappa,+,\mathrm{perf}}$ to the opens in $\mathfrak{U}$. The Čech-to-sheaf spectral sequence

$$E_2^{ij} = \check{H}^i(\mathfrak{U}, \mathcal{H}^j(\mathcal{X}'(\epsilon)^{\mathrm{perf}}, \omega^{\kappa,+,\mathrm{perf}})) \Rightarrow H^{i+j}(\mathcal{X}'(\epsilon)^{\mathrm{perf}}, \omega^{\kappa,+,\mathrm{perf}})$$

therefore degenerates at the E_2-page (in the almost category) and shows that for any $i > 0$,

$$\check{H}^i(\mathfrak{U}, \omega^{\kappa,+,\mathrm{perf}}) \overset{a}{=} H^i(\mathcal{X}'(\epsilon)^{\mathrm{perf}}, \omega^{\kappa,+,\mathrm{perf}}).$$

This reduces us to showing that the Čech-complex $\check{C}^{\bullet}(\mathfrak{U}, \omega^{\kappa,+,\mathrm{perf}})$ is almost acyclic, which by Corollary 4.2.1.2 agrees with the complex $\check{C}^{\bullet}(\mathfrak{U}, \mathfrak{w}^{\kappa,\mathrm{perf}})$.

To see that this complex is acyclic, recall from Lemma 11.1.1 that we can write

$$\mathfrak{w}^{\kappa,\mathrm{perf}} = (\varinjlim\nolimits_{F_{\mathrm{abs}}} \mathfrak{w}^{\kappa_n})^{\wedge} \quad \text{where } \kappa_n := \kappa^{p^n}.$$

In particular, we have $\mathfrak{w}^{\kappa,\mathrm{perf}}(U) = (\varinjlim_{F_{\mathrm{abs}}} \mathfrak{w}^{\kappa_n}(U))^{\wedge}$ for any $U \in \mathfrak{U}$, where $\wedge$ is the t-adic module completion. We therefore have a vertical directed system of horizontal complexes

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle F} & & \\
0 & \longrightarrow & \prod_i \mathfrak{w}^{\kappa^p}(U_i) & \longrightarrow & \prod_{i,j} \mathfrak{w}^{\kappa^p}(U_i \cap U_j) & \longrightarrow & \dots \\
 & & \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle F} & & \\
0 & \longrightarrow & \prod_i \mathfrak{w}^{\kappa}(U_i) & \longrightarrow & \prod_{i,j} \mathfrak{w}^{\kappa}(U_i \cap U_j) & \longrightarrow & \dots
\end{array}$$

By Proposition 11.1.2, we can now uniformly find $N > 0$ such that the cohomology $\check{H}^q_n := \check{H}^q(\mathfrak{U}, \mathfrak{w}^{\kappa_n})$ in the horizontal complex is annihilated by t^N for all q and for each n.

At this point, the argument in [48] carries over without changes: Let us write F for the absolute Frobenius on $\mathcal{O}_K$. Since the transition maps are F-linear in $\mathcal{O}_K$, from the point of view of the bottom row, the n-th line of this system is a module via $F^n : \mathcal{O}_K \to \mathcal{O}_K$. With this module structure, $\check{H}^q_n$ is already annihilated by t^{N/p^n}. But this means that the image of $\check{H}^q_n$ in $\check{H}^q_{n+m}$ is even annihilated by $t^{N/p^{n+m}}$. In the limit, this shows that the $\mathcal{O}_K = \varinjlim_F \mathcal{O}_K$-module $\check{H}^q_\infty := \varinjlim_n \check{H}^q_n$ is annihilated by t^{N/p^m} for every $m > 0$, and in particular is almost zero. By Lemma A.3.5, the complex is still almost exact after completion.

All in all, this shows that as desired we have for all $q \geq 1$:

$$H^q(\mathcal{X}'(\epsilon)^{\mathrm{perf}}, \omega^{\kappa,+,\mathrm{perf}}) \overset{a}{=} \check{H}^q(\mathfrak{U}, \mathfrak{w}^{\kappa,\mathrm{perf}}) = \check{H}^q(\mathfrak{U}, (\varinjlim\nolimits_F \mathfrak{w}^{\kappa_n})^{\wedge}) \overset{a}{=} 0. \qquad \square$$

11.2 Almost acyclicity in characteristic 0

Having completed the proof of the Almost Acyclicity Theorem in characteristic p, we now move on to the second part of the theorem. We recall the setup: Let K be a perfectoid field of characteristic 0 and let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a weight. Theorem 11.0.2.2 claims that the sheaf $\omega^{\kappa,+,\mathrm{perf}}$ on $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ is almost acyclic, i.e. that $H^q(\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a, \omega^{\kappa,+,\mathrm{perf}}) \overset{a}{=} 0$ for $q \geq 1$. We will deduce this from the case of characteristic p by way of the following two lemmas:

Fix a uniformiser t of $K^\flat$ such that $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t$. Recall that there is a canonical tilting isomorphism $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a = \mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ which induces an almost equality

$$\mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}/p \overset{a}{=} \mathcal{O}^+_{\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}}/t.$$

As before, we shall say that two elements $x \in \mathcal{O}_K$ and $y \in \mathcal{O}_{K^\flat}$ are congruent and write $x \equiv y$ if their images in $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t$ match up.

Lemma 11.2.1. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ and $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be two weights such that $\kappa \equiv \kappa^\flat$. Then inside $\mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}/p \overset{a}{=} \mathcal{O}^+_{\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}}/t$ we have a natural identification*

$$\omega^{\kappa,+,\mathrm{perf}}/p \overset{a}{=} \mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}/p[\kappa^{-1}] \overset{a}{=} \mathcal{O}^+_{\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}}/t[(\kappa^\flat)^{-1}] \overset{a}{=} \omega^{\kappa^\flat,+,\mathrm{perf}}/t.$$

Hence $\omega^{+,\kappa,\mathrm{perf}}/p$ depends only on $\kappa \bmod p$ and $\omega^{+,\kappa^\flat,\mathrm{perf}}/t$ only depends on $\kappa^\flat \bmod t$.

Proof. The almost equality in the middle is clear from $\kappa \equiv \kappa^\flat$.

To see the first equality, we first recall that for any perfectoid space X over a perfectoid field L with pseudo-uniformiser ϖ, the presheaf $\mathcal{O}^+_{\text{proét}}$ on the pro-étale site of X in the sense of [51] is a sheaf, and is almost acyclic on affinoid perfectoids by [51], Proposition 8.5.(iii). Consequently, for any affinoid perfectoid Y in $X_{\text{proét}}$, we have $\mathcal{O}^+_{\text{proét}}/p(Y) \overset{a}{=} \mathcal{O}^+(Y)/p$.

Applying this to the pro-étale $\mathbb{Z}_p^\times$-torsor of affinoid perfectoid spaces $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$, we conclude that we have an almost isomorphism

$$(\mathcal{O}^+_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}/p)^{\mathbb{Z}_p^\times} \overset{a}{=} \mathcal{O}^+_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}/p.$$

Applying Lemma 2.8.4 to the diagram of ringed spaces

$$\begin{tikzcd}
(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a, \mathcal{O}^+/p) \arrow[r] \arrow[d] & (\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a, \mathcal{O}^+/p) \arrow[d] \\
(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a, \mathcal{O}^+) \arrow[r] & (\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a, \mathcal{O}^+)
\end{tikzcd}$$

therefore gives the first almost equality. The third almost equality follows using the analogous diagram for the pro-étale $\mathbb{Z}_p^\times$-torsor $\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$. □

The proof of Theorem 11.0.2.2 is completed by the following Lemma:

Lemma 11.2.2. *Let X be a perfectoid space over a ϖ-adic perfectoid field K. Let V be a locally free $\mathcal{O}_X^+$-module on X. Then V/ϖ is almost acyclic if and only if V is almost acyclic.*

Proof. The long exact sequence associate to $0 \to V \xrightarrow{\varpi\cdot} V \to V/\varpi \to 0$ shows that V/ϖ is almost acyclic if V is. Conversely, if V/ϖ is almost acyclic, the long exact sequence of

$$0 \to V/\varpi \to V/\varpi^n \to V/\varpi^{n-1} \to 0$$

shows inductively that V/ϖ^n is almost acyclic for any $n \in \mathbb{N}$. Since X is perfectoid, we have $\mathcal{O}_X^+ \overset{a}{=} \varprojlim \mathcal{O}_X^+/\varpi^n$. Consequently, we see locally on any trivialising cover of the vector bundle V that we also have $V = \varprojlim V/\varpi^n$. We are therefore left to see that $\varprojlim V/\varpi^n$ is still almost acyclic. For this we first note that by [49], Lemma 3.18 applied to the basis B of affinoid perfectoid open subspaces of X on which V is free,

$$R^q \varprojlim V/\varpi^n = 0 \text{ for all } q > 0 \tag{33}$$

(here we are using the almost version of the Lemma which holds by the same proof. In fact, it is this almost version which is later used in [49] in the proof of Lemma 4.10.(v)).

In order to deduce the Theorem, we wish to use the Grothendieck spectral sequence for

$$\Gamma \circ \varprojlim = \varprojlim \circ \Gamma.$$

To see that the necessary technical conditions are satisfied, we recall that the functor $\varprojlim : \mathrm{Sh}^{\mathbb{N}} \to \mathrm{Sh}$ preserves injectives since it has an exact left adjoint given by sending a sheaf $\mathcal{F}$ to the constant system $(\mathcal{F})_{n\in\mathbb{N}}$. Similarly, the functor $\Gamma : \mathrm{Sh}^{\mathbb{N}} \to (\mathcal{O}_K^a\text{−modules})^{\mathbb{N}}$ has an exact left adjoint given by sending (M_n) to the system of locally constant sheaves $(\underline{M_n})$. We therefore do have two Grothendieck spectral sequence with respective second pages

$$H^i(X, R^j \varprojlim V/\varpi^n) \Rightarrow R^{i+j}(\Gamma \circ \varprojlim)((V/\varpi^n)_{n\in\mathbb{N}}),$$

$$R^i \varprojlim(H^j(X, V/\varpi^n)) \Rightarrow R^{i+j}(\Gamma \circ \varprojlim)((V/\varpi^n)_{n\in\mathbb{N}}).$$

By equation (33), the first spectral sequence degenerates at the second page and shows that

$$R^i(\Gamma \circ \varprojlim)((V/\varpi^n)_{n\in\mathbb{N}}) \overset{a}{=} H^i(X, V) \quad \text{for all } i \geq 0.$$

Since $H^j(X, V/\varpi^n) \overset{a}{=} 0$ for $j > 0$, the second spectral sequence also degenerates and shows

$$H^1(X, V) = R^1 \varprojlim \Gamma(V/\varpi^n), \quad \text{and } H^i(X, V) \overset{a}{=} 0 \text{ for all } i > 1.$$

But the short exact sequence from the beginning of the proof shows that the maps $\Gamma(V/\varpi^n) \to \Gamma(V/\varpi^{n-1})$ are almost surjective, thus also $H^1(X, V) = R^1 \varprojlim \Gamma(V/\varpi^n) \overset{a}{=} 0$. □

proof of Theorem 11.0.2.2. Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be any weight with $\kappa \equiv \kappa^\flat \bmod p$. By Theorem 11.0.2.1, the sheaf $\omega^{\kappa^\flat,+,\mathrm{perf}}$ is almost acyclic. By Lemma 11.2.2, the same is true for $\omega^{\kappa^\flat,+,\mathrm{perf}}/t$. By Lemma 11.2.1, this almost equals $\omega^{\kappa,+,\mathrm{perf}}/p$ as a sheaf on $|\mathcal{X}'(\epsilon)^{\mathrm{perf}}| = |\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a|$. The result thus follows from applying the other direction of Lemma 11.2.2. □

11.3 Corollaries

We note the following immediate Corollary of the Acyclicity Theorem 11.0.2:

Corollary 11.3.1. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ and $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be weights with $\kappa \equiv \kappa^\flat \bmod p$. Let $0 \leq \epsilon \leq \min(\epsilon_{\kappa_1}, \epsilon_{\kappa_2})$. Then $\mathcal{O}^+(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)/p \overset{a}{=} \mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})/t$ identifies*

$$M_\kappa^{+,\mathrm{perf}}(\epsilon)/p \overset{a}{=} \mathcal{O}(\omega^{\kappa,+,\mathrm{perf}}/p) \overset{a}{=} \mathcal{O}(\omega^{\kappa^\flat,+,\mathrm{perf}}/t) \overset{a}{=} M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon)/t.$$

Proof. The long exact sequences of multiplication by p on $\omega^{\kappa,+,\mathrm{perf}}$ and by t on $\omega^{\kappa^\flat,+,\mathrm{perf}}$ give short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M_\kappa^{+,\mathrm{perf}}(\epsilon)/p \arrow[r] \arrow[d, dotted] & \mathcal{O}(\omega^{\kappa,+,\mathrm{perf}}/p) \arrow[r] \arrow[d, equal, "a"] & H^1(\omega^{\kappa,+,\mathrm{perf}})[p] \arrow[r] & 0 \\ 0 \arrow[r] & M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon)/t \arrow[r] & \mathcal{O}(\omega^{\kappa^\flat,+,\mathrm{perf}}/t) \arrow[r] & H^1(\omega^{\kappa^\flat,+,\mathrm{perf}})[t] \arrow[r] & 0. \end{tikzcd}$$

which are connected by the almost inequality in the middle from Lemma 11.2.1. By Theorem 11.0.2, the two terms on the right are both almost zero. This gives the desired almost isomophism $M_\kappa^{+,\mathrm{perf}}(\epsilon)/p \overset{a}{=} M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon)/t$. Since the left horizontal maps and the map from Lemma 11.2.1 all commute with the maps into $\mathcal{O}^+(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)/p \overset{a}{=} \mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})/t$, so does the dotted arrow. □

Lemma 11.3.2. *Assume $K = \mathbb{Q}_p^{\mathrm{cyc}}$, $K^\flat = \mathbb{F}_p((t^{1/(p-1)p^\infty}))$ and $\kappa^\flat = \overline{\kappa}$, $\kappa = \kappa^{\flat\sharp}$.*

1. *The map from Corollary 11.3.1.1 can be explicitly described as the honest module map $s : M_{\overline{\kappa}}^{+,\mathrm{perf}}(\epsilon)/t \to M_\kappa^{+,\mathrm{perf}}(\epsilon)/p$ obtained from reducing $\sharp$ mod p.*

2. *For any $f \in M_{\overline{\kappa}}^{+,\mathrm{perf}}(\epsilon)$ and any $l \in \mathbb{Z}[1/p]_{>0}$ with $l \leq 1$, and any lift $\mathfrak{f} \in \mathcal{O}(\mathfrak{w}_{\infty,\infty,l})$ of f with specialisation $\mathfrak{f}_\kappa$ at κ, we have $s(f) \equiv \mathfrak{f}_\kappa \bmod p^{1-l}$ inside $M_\kappa^{+,\mathrm{perf}}(\epsilon)$.*

Proof. The first part is clear from the definitions since for any perfectoid K-algebra A, the map $\sharp : A^{\flat\circ} \to A^\circ$ is the composition of $[-] : A^{\flat\circ} \to W(A^{\flat\circ})$ with the specialisation $W(A^{\flat\circ}) \to A^\circ$ induced by base-change along $\overline{\kappa}^\sharp : W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$.

For the second it suffices to verify that the described map commutes with the inclusions into the spaces $\mathcal{O}^+(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)/p^{1-l} \overset{a}{=} \mathcal{O}^+(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})/t^{1-l}$. For this recall that the lift $\mathfrak{f}$ lives in $\mathcal{O}(\mathfrak{IG}_{\infty,\infty,\infty,l})$. The forms f and $\mathfrak{f}_\kappa$ are then obtained by base-change along the specialisation maps in weight space given by the top of the following diagram

$$\begin{tikzcd}[column sep=small] & \mathbb{Z}_p[[(1+T)^{1/p^\infty}]]\langle p/S^l\rangle \arrow[dl, "\overline{\kappa}"'] \arrow[dr, "\overline{\kappa}^\sharp"] & \\ \mathbb{F}_p[[t^{1/(p-1)p^\infty}]] \arrow[dr] & & \mathbb{Z}_p^{\mathrm{cyc}} \arrow[dl] \\ & \mathbb{F}_p[[t^{1/(p-1)p^\infty}]]/t^{1-l} = \mathbb{Z}_p^{\mathrm{cyc}}/p^{1-l} & \end{tikzcd}$$

where $\overline{\kappa}^\sharp$ sends $S_k \mapsto x_n := (t^{1/p^n}+1)^\sharp - 1$, and where Notation 6.1.16 gives a sense to p^{1-l}. We note that $|x_0| = |p|$ and thus $|x_n| = |p|^{1/p^n}$ for all n, for example by Lemma 6.1.17.

It is clear from the isomorphism $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]/t = \mathbb{Z}_p^{\mathrm{cyc}}/p$ induced by the identification $\mathbb{Q}_p^{\mathrm{cyc}\flat} = \mathbb{F}_p((t^{1/(p-1)p^\infty}))$, and from $0 < l \le 1$, that this diagram commutes when restricted to $\mathbb{Z}_p[[(1+T)^{1/p^\infty}]]$. To see that the whole diagram commutes, it therefore suffices to consider the images of p/S^l: As it is sent to 0 by $\overline{\kappa}$, we have to see that it vanishes when sent along the right hand side of the diagram. To this end, let $m \gg 0$ be such that $l \in p^{-m}\mathbb{Z}$, then we may replace p/S^l by $p/S_m^{lp^m}$. The map $\overline{\kappa}^\sharp$ sends this to $p/x_m^{lp^m}$. We now have

$$|p/x_m^{lp^m}| = |p|/|p|^{lp^m/p^m} = |p|/|p|^l = |p|^{1-l}.$$

Consequently, $p/x_m^{lp^m} \in p^{1-l}\mathbb{Z}_p^{\mathrm{cyc}}$ as desired, and we see that indeed the diagram commutes. Thus so do the corresponding specialisations of $\mathfrak{f}$. □

As a second application of the Almost Acyclicity Theorem, we shall now complete the proof of the base change result Proposition 4.3.3. We recall that the remaining case was that K is a perfectoid field extension of either $\mathbb{F}_p((t))$ or $\mathbb{Q}_p^{\mathrm{cyc}}$.

Proof of Proposition 4.3.3 for perfectoid K. We give the proof in the case that K is a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$, the case that K is of characteristic p is completely analogous.

We start with the perfectoid modular forms: Let $C^\bullet$ be the Čech complex of some open cover of $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ by rational subspaces on which $\omega^{\kappa,+,\mathrm{perf}}$ is trivial. Then $M_\kappa^{+,\mathrm{perf}}(\epsilon) = H^0(C^\bullet)$ and $M_{\kappa,L}^{+,\mathrm{perf}}(\epsilon) = H^0(C^\bullet\hat{\otimes}\mathcal{O}_L)$. By a Čech-to-sheaf spectral sequence,

$$H^*(C^\bullet) \overset{a}{=} H^*(\omega^{\kappa,+,\mathrm{perf}}),$$

which is almost zero by Theorem 11.0.2.2. In particular, the cohomology of $C^\bullet$ has bounded p-torsion. Then by Lemma A.3.6, we have

$$M_{\kappa,L}^{+,\mathrm{perf}}(\epsilon) = H^0(C^\bullet\hat{\otimes}\mathcal{O}_L) = H^0(C^\bullet)\hat{\otimes}\mathcal{O}_L = M_\kappa^{+,\mathrm{perf}}(\epsilon)\hat{\otimes}\mathcal{O}_L.$$

The case of $M^+_{\kappa,L}(\epsilon)$ follows from this: By comparison to the perfectoid case using the traces of Proposition 2.2.6, the map $M_\kappa^+(\epsilon)\hat{\otimes}\mathcal{O}_L \to M^+_{\kappa,\mathcal{O}_L}(\epsilon)$ is injective with cokernel killed by p^n for some n. Moreover, $f \in M_\kappa^+(\epsilon)/p \hookrightarrow M_\kappa^{+,\mathrm{perf}}(\epsilon)/p$ is injective, and this stays true after tensoring with $\mathcal{O}_L$. Since for any $f \in M^+_{\kappa,\mathcal{O}_L}(\epsilon)$ we have $p^n f \in M_\kappa^+(\epsilon)\hat{\otimes}\mathcal{O}_L$ and $f \in M^{+,\mathrm{perf}}_{\kappa,\mathcal{O}_L}(\epsilon) = M_\kappa^{+,\mathrm{perf}}(\epsilon)\hat{\otimes}\mathcal{O}_L$ by the first part, this shows that already $f \in M_\kappa^+(\epsilon)\hat{\otimes}\mathcal{O}_L$. □

12 The overconvergent tilting isomorphism

We now finally put all ingredients together and prove our main result, the overconvergent tilting isomorphism for p-adic modular forms. We begin with the perfectoid version.

12.1 The perfectoid overconvergent tilting isomorphism

Recall that in Theorem 5.2.2, we proved a first version of an overconvergent tilting isomorphism for perfectoid modular forms. We are now equipped to prove a version of this Theorem which allows greater flexibility in the choice of weights on the p-adic side.

Theorem 12.1.1. *Let K be any perfectoid field. Let t be a pseudo-uniformiser of $K^\flat$ such that $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t$. Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ be a family of weights such that $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ for all $n \in \mathbb{Z}_{\geq 0}$. Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be the weight corresponding to this family under the isomorphism $\mathcal{O}_{K^\flat}^\times = \varprojlim_{x\mapsto x^p}(\mathcal{O}_K/p)^\times$. Then:*

1. *For any n, the absolute Frobenius on $\mathcal{O}^{+a}_{\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a}/p$ restricts to a map*
$$\omega^{\kappa_{n+1},+,\mathrm{perf}}/p \to \omega^{\kappa_n,+,\mathrm{perf}}/p, \quad f \mapsto f^p.$$
2. *There is a natural isomorphism of sheaves on $|\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}| = |\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a|$*
$$\omega^{\kappa^\flat,+,\mathrm{perf}} \overset{a}{=} \varprojlim_{f\mapsto f^p} \omega^{\kappa_n,+,\mathrm{perf}}/p.$$
3. *For any n, the absolute Frobenius on $\mathcal{O}^{+a}(\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)/p$ restricts to a map*
$$F : M^{+,\mathrm{perf}}_{\kappa_{n+1}}(\epsilon)/p \to M^{+,\mathrm{perf}}_{\kappa_n}(\epsilon)/p, \quad f \mapsto f^p.$$
4. *There is a natural $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t$-linear almost isomorphism*
$$M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon) \overset{a}{=} \varprojlim_F M^{+,\mathrm{perf}}_{\kappa_n}(\epsilon)/p.$$

Proof. Let us write $\kappa_n^\flat := (\kappa^\flat)^{1/p^n}$, then $\kappa_n^\flat \equiv \kappa_n$. For part 1, we recall from Lemma 3.6.4 that $F_{\mathrm{abs}}^*\mathfrak{w}^{\kappa_{n+1}^\flat} = \mathfrak{w}^{\kappa_n^\flat}$. Pulling back to $\mathcal{O}^+/t$-modules, this gives by adjunction a map

$$\omega^{\kappa_{n+1}^\flat,+,\mathrm{perf}}/t = F_{\mathrm{abs}*}\omega^{\kappa_n^\flat,+,\mathrm{perf}}/t^p = \omega^{\kappa_n^\flat,+,\mathrm{perf}}/t^p \twoheadrightarrow \omega^{\kappa_n^\flat,+,\mathrm{perf}}/t$$

where in the second step we use that the absolute Frobenius is the identity on topological spaces. Part 1 follows from this since by Lemma 11.2.1, we have

$$\omega^{\kappa_n^\flat,+,\mathrm{perf}}/t \overset{a}{=} \omega^{\kappa_n,+,\mathrm{perf}}/p.$$

For part 2, we use that for any perfectoid space Y over $K^\flat$ we have $\mathcal{O}_Y^+ = \varprojlim \mathcal{O}_Y^+/t^{p^n} = \varprojlim_{x\mapsto x^p} \mathcal{O}_Y^+/t$. Since $\omega^{\kappa^\flat,+,\mathrm{perf}}$ is an invertible $\mathcal{O}^+$-module, it follows locally that also

$$\omega^{\kappa^\flat,+,\mathrm{perf}} = \varprojlim \omega^{\kappa^\flat,+,\mathrm{perf}}/t^{p^n} = \varprojlim_{x\mapsto x^p} \omega^{\kappa_n^\flat,+,\mathrm{perf}}/t = \varprojlim_{x\mapsto x^p} \omega^{\kappa_n,+,\mathrm{perf}}/p$$

where the second and third step are the two displayed isomorphisms above, respectively.

Part 3 follows from part 1 and part 4 follows from part 2 on taking global sections by the Acyclicity Theorem, or more precisely by its Corollary 11.3.1. □

12.2 The main theorem

We will now finally put together the ingredients prepared in the last sections and prove our main theorem, the tilting isomorphism of overconvergent p-adic modular forms.

Theorem 12.2.1. *Let K be a perfectoid extension of $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of weights such that $\kappa_{n+1}^p \equiv \kappa_n$ for all n and $|T_{\kappa_0}| \geq |p|$ and $|T_{\kappa_1}| > |p|^{1/(p-1)}$. Let $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$ be the weight corresponding to $(\kappa_n)_{n\in\mathbb{Z}_{\geq 0}}$ under $\mathcal{O}_{K^\flat}^\times = \varprojlim_{x\mapsto x^p}(\mathcal{O}_K/p)^\times$. Let either $\epsilon = 0$, or $\epsilon = p^{-c}\epsilon_{\kappa_0}$ for some $c \in \mathbb{Z}_{\geq 1}$ (here $\epsilon_{\kappa_0} > 0$ is defined in Definition 3.3.6).*

Let $l := 3/p^{r_0+c-2}(p-1) + 1/p^c$, or $l = 0$ if $\epsilon = 0$. The conditions imply that $0 \leq l < 1$.

1. *For any n, the morphism $F : M_{\kappa_{n+1}}^{+,\mathrm{perf}}(\epsilon)/p^{1-l} \to M_{\kappa_n}^{+,\mathrm{perf}}(\epsilon)/p^{1-l}$ restricts to a map*
$$M^+_{\kappa_{n+1},\Gamma_0(p^{n+1})}(\epsilon)/p^{1-l} \to M^+_{\kappa_n,\Gamma_0(p^n)}(\epsilon)/p^{1-l}, \quad f \mapsto f^p$$
which is equivariant for the action of the integral Hecke algebra.

2. *There is a canonical $\mathcal{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p^{1-l}$-linear almost isomorphism*
$$M^+_{\kappa^\flat}(\epsilon) \overset{a}{=} \varprojlim_{f\mapsto f^p} M^+_{\kappa_n,\Gamma_0(p^n)}(\epsilon)/p^{1-l} \overset{a}{=} \varprojlim_{f\mapsto f^{(p)}} M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}$$
which is equivariant for the action of the integral Hecke algebra.

3. *For any n, projection from the inverse system induces an almost isomorphism*
$$M^+_{\kappa^\flat}(\epsilon)/t^{(1-l)p^n} \overset{a}{\xrightarrow{\sim}} M^+_{\kappa_n,\Gamma_0(p^n)}(\epsilon)/p^{1-l} = M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}, \quad f \mapsto f^{1/p^n}$$
which is equivariant for the action of the integral Hecke algebra.

Remark 12.2.2. The role of the conditions on $|T_{\kappa_1}|$ and $|T_{\kappa_0}|$ is twofold: First, they ensure that the κ_n converge to the boundary of weight space, since by Lemma 6.1.17, they imply $|T_{\kappa_{n+1}}|^p = |T_{\kappa_n}|$ for all n. Second, they ensure that the condition on ϵ only depends on κ_0.

Before we prove the Theorem, we note a few ways in which it can be further enhanced:

Remark 12.2.3. In analogy to part 1 of the perfectoid overconvergent tilting equivalence, one can upgrade the Theorem to a version that identifies the entire sheaf of modular forms. Namely, there is a canonical isomorphism of sheaves of $\mathcal{O}_K^a$-modules on $\mathfrak{X}'(\epsilon)$
$$\mathfrak{w}^{\overline{\kappa}} = \varprojlim_{f\mapsto f^p} \phi^{n*}\mathfrak{w}^{\kappa_n}/p^{1-l} = \varprojlim_{f\mapsto f^{(p)}} i^{n*}\mathfrak{w}^{\kappa_n}/p^{1-l}$$
where the maps $i^n, \phi^n : \mathfrak{X}(p^{-n}\epsilon) \rightrightarrows \mathfrak{X}(\epsilon)$ are the restriction and Frobenius lift, respectively, and the second isomorphism is the one from Proposition 2.4.1. In particular, one might a posteriori take the above isomorphism as the *definition* of t-adic modular forms.

We note that this version uses formal sheaves, in contrast to the perfectoid version that uses analytic sheaves: This is because one needs to use traces, which we are only able to define for formal schemes. It also allows us to canonically identify $|\mathfrak{X}'(\epsilon)| = |\mathfrak{X}'(\epsilon)/t^{1-\epsilon}| = |\mathfrak{X}(\epsilon)/p^{1-\epsilon}| = |\mathfrak{X}(\epsilon)|$, whereas the natural map $|\mathcal{X}'(\epsilon)| \to |\mathcal{X}(\epsilon)|$ is a cover of infinite degree.

The proof is essentially the same as the proof of the one for the stated Theorem, by replacing $\mathfrak{X}(\epsilon)$ with some open $U \subseteq \mathfrak{X}(\epsilon)$, and replacing the canonical section with the morphism $\natural : \mathfrak{w}^{\overline{\kappa}} \to \mathfrak{w}^{\kappa}$ from Remark 9.4.3. Carrying this out over a general field K requires the more general canonical lift discussed in the subsequent remark.

We note that in contrast to the perfectoid situation, the above statement does not immediately imply the Theorem, since the sheaves $\mathfrak{w}^{\kappa_n}$ are not acyclic. This can be remedied by noting that, as the $\natural$-map shows, at least on generically affinoid open subspaces one can replace the sheaf quotient on the right hand side by presheaf quotients.

Remark 12.2.4. The restriction on $\epsilon > 0$ to be included in the discrete set of radii described in Theorem 12.2.1 is just an artefact of our proof, and one should be able to get a statement for more general $0 \leq \epsilon \leq p^{-c}\epsilon_\kappa$ by the same strategy of proof. For this, however, one needs to base change the constructions from §6 to §9 along the map $\mathbb{Z}_p[[(1+T)^{1/p^\infty}]] = W(\mathbb{F}_p[[t^{1/p^\infty}]]) \to W(\mathcal{O}_{K^\flat})$ sending $[t] \mapsto [t^\epsilon]$, thereby replacing the radius variable r by ϵ.

One can then reprove the main results of sections §§7, 8 and 9 in this setting: The sous-perfectoidness just follows by base-change. The results from §7 are more work: Lemma 6.5.4 still holds, by a version of the criterion Proposition 4.1.4 for integral perfectoid rings over $W(\mathcal{O}_{K^\flat})\langle p/T\rangle$. Proposition 7.1.1 does not simply follow by base-change, but the same proof works. Proposition 7.3.4 does follow by base-change.

Resetting the structure map to the natural map $\mathbb{Z}_p[[(1+T)^{1/p^\infty}]] \to W(\mathcal{O}_{K^\flat})$ defined by sending $[t]$ to $[t]$, one can by these preparations define a bundle of modular forms like in §9.1. Proposition 9.2.2 can then be seen in the same way. Putting everything together, one then obtains a generalisation of Theorem 9.4.1 for general base rings and general ϵ.

However, since we do not have an application in mind for any additional flexibility in ϵ, we shall not carry this out in detail in the present work.

Proof of Theorem 12.2.1. The plan is to start by proving part 3, and then deduce 1 and 2.

As a first step, we prove that the $\natural$-map of Definition 9.4.2 combines with the isomorphism $M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t \overset{a}{=} M^{+,\mathrm{perf}}_{\kappa}(\epsilon)/p$ of Proposition 11.3.1 to induce an $\mathcal{O}_{K^\flat}/t = \mathcal{O}_K/p$-linear map

$$\natural : M^{+}_{\kappa^\flat}(\epsilon)/t \xrightarrow{a} M^{+}_{\kappa}(\epsilon)/p$$

which becomes a Hecke-equivariant almost isomorphism mod p^{1-l}. Here and in the following, we set $\kappa := \kappa_0$ and work on the weight disc determined by κ.

We first make a few reduction steps: By combining the perfectoid tilting equivalence Theorem 12.1.1 with Corollary 4.3.4.3, it suffices to construct the above map after passing to any perfectoid extension of K. We may therefore without loss of generality assume that κ admits arbitrary p-th roots κ^{1/p^n}, and we fix such a choice. We do not replace κ_n by κ^{1/p^n}, we just need this data for the construction of $\natural$ and will forget about the κ^{1/p^n} afterwards.

Second, we may assume without loss of generality that $\kappa^\flat$ is the weight κ' determined by $(\kappa^{1/p^n})_{n\in\mathbb{Z}_{\geq 0}}$. This is because first, by Lemma 6.1.17, we have $|T_{\kappa^{1/p}}| > |p|^{1/(p-1)}$ as required. Second, we have $\kappa' \equiv \kappa$ in $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t$ and thus $\kappa' \equiv \kappa^\flat \bmod t$. By Proposition 10.5.2, we thus have a Hecke-equivariant identification inside $\mathcal{O}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})/t$:

$$M^{+}_{\kappa^\flat}(\epsilon)/t = M^{+}_{\kappa'}(\epsilon)/t.$$

Third, the assumption that $\kappa_n \not\equiv 1$ ensures that $\kappa^\flat \neq 1$ and we may therefore relate $\kappa^\flat$ to the canonical weight $\overline{\kappa} : \mathbb{Z}_p^\times \to \mathbb{F}_p[[t^{1/p^\infty}]]^\times$, $1 \mapsto 1+t$ as usual: Let $\iota : \mathbb{F}_p((t^{1/p^\infty})) \hookrightarrow K^\flat$ be such that $\kappa^\flat = \iota \circ \overline{\kappa}$. By Proposition 4.3.3, we then have

$$M^{+}_{\kappa^\flat}(\epsilon) = M^{+}_{\kappa^\flat}(p^{-c}\epsilon_\kappa) = M^{+}_{\overline{\kappa}}(p^{-(r_0+1+c)})\hat{\otimes}_\iota \mathcal{O}_{K^\flat}.$$

Let us set $r := r_0 + c$, then in particular $M^{+}_{\kappa^\flat}(\epsilon)/t = M^{+}_{\overline{\kappa}}(p^{-r-1}) \otimes_\iota \mathcal{O}_{K^\flat}/t$. By $\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t$-linearity, we are reduced to constructing an ι-semilinear map $M^{+}_{\overline{\kappa}}(p^{-r-1}) \to M^{+}_{\kappa}(\epsilon)/p$.

We note that we now consider two different copies of $\mathbb{F}_p((t^{1/p^\infty})) \hookrightarrow K^\flat$, which do two different things for us: The first inclusion comes from the $\mathbb{Q}_p^{\mathrm{cyc}}$-algebra structure on K and our identification $\mathbb{F}_p((t^{1/(p-1)p^\infty})) = \mathbb{Q}_p^{\mathrm{cyc}\flat}$, it therefore fixes the untilt $K|\mathbb{Q}_p^{\mathrm{cyc}}$ of $K^\flat$. Let us denote this map by j. The embedding $\iota : \mathbb{F}_p((t^{1/p^\infty})) \hookrightarrow K^\flat$ instead fixes the weight. When we pass to Witt vectors and untilts, these two embeddings fit into a commutative diagram

$$\begin{tikzcd}
 & \mathcal{O}_K & \mathbb{Z}_p^{\mathrm{cyc}} \arrow[l] \\
\mathbb{Z}_p[[(1+T)^{1/p^\infty}]] \arrow[r, "W(\iota)"] \arrow[d] \arrow[ur, dotted, "(\kappa^{1/p^n})_{n\in\mathbb{Z}_{\geq 0}}"] & W(\mathcal{O}_{K^\flat}) \arrow[u, "\theta"] \arrow[d] & W(\mathbb{F}_p[[t^{1/p^\infty}]]) \arrow[l, "W(j)"'] \arrow[u, "\theta"] \arrow[d] \\
\mathbb{F}_p[[t^{1/p^\infty}]] \arrow[r, "\iota"] & \mathcal{O}_{K^\flat} & \mathbb{F}_p[[t^{1/p^\infty}]]. \arrow[l, "j"']
\end{tikzcd}$$

Since we assume that $(\kappa^{\flat/p^n})^\sharp = \kappa^{1/p^n}$, the diagram tells us that we recover our family of weights $(\kappa^{1/p^n})_{n\in\mathbb{Z}_{\geq 0}}$ as the point of perfected weight space defined by $\theta\circ W(\iota)$.

By Definition 9.4.2, the map $\natural : M^+_{\overline{\kappa}}(p^{-r-1}) \to M^+_\kappa(\epsilon)$ can now be described as follows: By our assumptions on κ_0 and κ_1, the sequence $(\kappa^{1/p^n})_{n\in\mathbb{N}}$ defines a point of $\mathcal{W}_{\infty,1}$. We would like to extend modular forms in $M^+_{\overline{\kappa}}(p^{-r-1})$ to families over "$\mathfrak{W}_{\infty,l}$". However, l might not be in $\mathbb{Z}[1/p]$, it is not clear that this is well-defined. We therefore use that $\mathbb{Z}[1/p]\subseteq\mathbb{Q}$ is dense and approximate: Let $1 > l_1 \geq l_2 \geq \dots$ be a sequence in $\mathbb{Z}[1/p]_{>0}$ that converges to $l = 3/p^{r-2}(p-1) + 1/p^{r-r_0}$. For the moment, let us work with l_d instead of l.

Since $l_d \geq l = 3/p^{r-2}(p-1) + 1/p^{r-r_0}$, we can now apply Theorem 9.4.1.2 to find a canonical lift of $f \in M^+_{\overline{\kappa}}(p^{-r-1}) = \mathcal{O}(\omega^+_{r,\infty,\infty})$ via the map $[-]_r$ to an integral family $\mathfrak{f} := [f]_r \in \mathcal{O}(\omega^+_{r,\infty,l_d})$ over $\mathfrak{W}_{\infty,l_d}$. We then specialise $\mathfrak{f}$ at $(\kappa^{1/p^n})_{n\in\mathbb{Z}_{\geq 0}}$ to obtain a modular form $f^\natural := \mathfrak{f}_\kappa \in M^+_\kappa(\epsilon)$. Here the radius ϵ appears since specialisation at κ sends the condition $|\mathrm{Ha}| \geq |S^{1/p^{r+1}}|$ that we impose on $\mathcal{X}_{r,\infty,l}$ to $|\mathrm{Ha}| \geq |T_\kappa|^{1/p^{r+1}} = |p|^\epsilon$ where we recall that by definition, $|T_\kappa|^{1/p^{r_0+1}} = |p|^{\epsilon_\kappa}$ and $r = r_0 + c$ and $\epsilon = p^{-c}\epsilon_\kappa$.

At this point, we have constructed a map $M^+_{\kappa^\flat}(\epsilon)/t \to M^+_\kappa(\epsilon)/p$ sending $f \mapsto f^\natural$.

Next, we claim that mod p^{1-l_d}, this map is compatible with the map $M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t \to M^{+,\mathrm{perf}}_\kappa(\epsilon)/p$ from Theorem 12.1.1 in the sense that the following diagram commutes:

$$\begin{tikzcd}
M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t^{1-l_d} \arrow[r, "\sharp", "\sim"'] & M^{+,\mathrm{perf}}_\kappa(\epsilon)/p^{1-l_d} \\
M^{+}_{\kappa^\flat}(\epsilon)/t^{1-l_d} \arrow[r, "\natural"] \arrow[u, hook] & M^{+}_\kappa(\epsilon)/p^{1-l_d}. \arrow[u, hook]
\end{tikzcd} \tag{34}$$

For this it suffices again by Proposition 4.3.3 to prove compatibility with the map

$$M^{+,\mathrm{perf}}_{\overline{\kappa}}(p^{-r-1}) \to M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t^{1-l_d} \to M^{+,\mathrm{perf}}_\kappa(\epsilon)/p^{1-l_d}$$

from Corollary 11.3.1.1. But by Lemma 11.3.2.2, this can be described as choosing *any* lift of f and specialising at κ. For $f \in M^+_{\overline{\kappa}}(p^{-r-1})$, we can thus take the lift $[f]_r$, as desired.

At this point, we have constructed the commutative diagram (35). It moreover follows immediately from the construction that $\natural$ is Hecke-equivariant, since f and its image $\natural(f)$ are specialisations of the same family of modular forms $\mathfrak{f}$, and specialisation is Hecke equivariant.

By the commutative algebra Lemma 12.2.5 below, we see in the inverse limit over d of the diagrams (34) that we also have a commutative diagram in the

$$\begin{tikzcd}
M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t^{1-l} \arrow[r, "\sharp", "\sim"'] & M^{+,\mathrm{perf}}_\kappa(\epsilon)/p^{1-l} \\
M^{+}_{\kappa^\flat}(\epsilon)/t^{1-l} \arrow[r, "\natural"] \arrow[u, hook] & M^{+}_\kappa(\epsilon)/p^{1-l}. \arrow[u, hook]
\end{tikzcd} \tag{35}$$

and the bottom map is still Hecke-equivariant as a morphism in the almost category.

Our next goal is to show that the bottom map $\natural$ in diagram (35) is an almost isomorphism. For this we use the commutative algebra Lemma 12.2.6 below: Since it is clear from (35) that $\natural$ is almost injective, it suffices to show that the cokernel of $\natural$ is almost annihilated by p^δ for some $1-l > \delta > 0$. To see the latter, we use that by construction of $\natural$, the diagram

$$\begin{tikzcd}
t^\delta M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t^{1-l} \arrow[r, "\sharp", "\sim"'] \arrow[d, "\mathrm{tr}"] & p^\delta M^{+,\mathrm{perf}}_{\kappa}(\epsilon)/p^{1-l} \arrow[d, "\mathrm{tr}"] \\
M^{+}_{\kappa^\flat}(\epsilon)/t^{1-l} \arrow[r, "\natural"] & M^{+}_{\kappa}(\epsilon)/p^{1-l}
\end{tikzcd}$$

commutes, where $\delta := 3\epsilon p^3/(p-1)$. Since $\mathrm{tr} : p^\delta M^{+,\mathrm{perf}}_{\kappa}(\epsilon) \to M^{+}_{\kappa}(\epsilon)$ restricts to the identity on $p^\delta M^{+}_{\kappa}(\epsilon)$ by Proposition 2.2.6, the image of the right vertical map in (12.2) contains $p^\delta M^{+}_{\kappa}(\epsilon)/p$, and thus the same is true for $\natural$.

For the Lemma to apply, it therefore suffices to see that $\delta < 1-l$, or equivalently

$$l+\delta = \frac{3}{p^{r-2}(p-1)} + \frac{1}{p^{r-r_0}} + \frac{3\epsilon p^3}{(p-1)} < 1. \tag{36}$$

We note that this in particular implies $l<1$, as claimed in the Theorem. To see that (36) holds, we first recall that

$$\delta = \frac{6}{3\epsilon p^3}(p-1) \le \frac{3}{p^{r-2}(p-1)}.$$

This is because $|T_\kappa| \ge |p|$ implies $|p|^{1/p^{r_0+1}} \le |T_\kappa|^{1/p^{r_0+1}} = |p|^{\epsilon_\kappa}$, where the last equality is the definition of ϵ_κ, and consequently, $1/p^{r_0+1} \ge \epsilon_\kappa$, and $1/p^{r+1} = 1/p^{r_0+c+1} \ge p^{-c}\epsilon_\kappa = \epsilon$.

We thus have

$$l+\delta \le \frac{6}{p^{r-2}(p-1)} + \frac{1}{p^c} \le \frac{6}{p^{r_0-1}(p-1)} + \frac{1}{p},$$

where the last step uses $r_0 = 5$ and the last displayed term equals $3/8 + 1/2 < 1$. For $p=3$, we have $r_0 = 3$ and the same term is $1/3+1/3<1$. For $p>3$, it is smaller than for $p=3$.

Thus all conditions of Lemma 12.2.6 are satisfied, and we obtain the desired Hecke-equivariant isomorphism

$$M^{+}_{\kappa^\flat}(\epsilon)/t^{1-l} \overset{a}{=} M^{+}_{\kappa}(\epsilon)/p^{1-l}.$$

By (35) and Theorem 12.2.1, we may see this as an equality inside $\mathcal{O}^{+a}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})/t^{1-l}$.

To deduce the more general statement of part 3 of the Theorem, we apply our discussion so far to $\kappa^\flat_n := (\kappa^\flat)^{1/p^{n-1}}$ and κ_n. Since $|T_{\kappa_{n+1}}|^p = |T_{\kappa_n}|$, we have $\epsilon_{\kappa_n} = p^{-n}\epsilon_\kappa$, and thus the weight κ_n satisfies the assumptions of the Theorem for ϵ replaced by $p^{-n}\epsilon$. In particular, we may leave c unchanged, and thus also l. By applying our construction so far, we obtain an isomorphism

$$M^{+}_{\kappa^\flat_n}(p^{-n}\epsilon)/t^{1-l} \xrightarrow{a} M^{+}_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}.$$

Precomposing with the inverse of base change along $F^n_{\mathrm{abs}} : \mathcal{O}_{K^\flat} \to \mathcal{O}_{K^\flat}$ (Lemma 3.6.4.2) on the left and postcomposing with the Atkin–Lehner isomorphism (Proposition 2.4.1.2) on the right, we get the desired $F^{-n}_{\mathrm{abs}} : \mathcal{O}_{K^\flat}/t^{(1-l)p^n} \xrightarrow{\sim} \mathcal{O}_{K^\flat}/t^{1-l} = \mathcal{O}_K/p^{1-l}$-linear isomorphism

$$M^{+}_{\kappa^\flat}(\epsilon)/t^{(1-l)p^n} \overset{a}{\underset{\sim}{\to}} M^{+}_{\kappa_n,\Gamma_0(p^n)}(\epsilon)/p^{1-l} \tag{37}$$

which is instead compatible with the isomorphism $M^{+,\mathrm{perf}}_{\kappa^\flat}(\epsilon)/t^{(1-l)p^n} \overset{a}{\underset{\sim}{\to}} M^{+,\mathrm{perf}}_{\kappa_n}(\epsilon)/p^{1-l}$ from Theorem 12.2.1.2. As the latter is given by $f \mapsto f^{1/p^n}$ inside $\mathcal{O}^{+a}(\mathcal{X}'_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}})/t^{1-l}$, the morphism (37) has the same description. This finishes the proof of part 3 of the Theorem.

Part 1 now follows from 3 by completing to a commutative diagram in the almost category

$$\begin{tikzcd}
M^{+}_{\kappa,\Gamma_0(p^{n+1})}(\epsilon)/p^{1-l} \arrow[r, dotted, "f\mapsto f^p"] & M^{+}_{\kappa^\flat,\Gamma_0(p^n)}(\epsilon)/p^{1-l} \\
M^{+}_{\kappa^\flat}(\epsilon)/t^{(1-l)p^{n+1}} \arrow[r] \arrow[u, "\wr", "f\mapsto f^{1/p^{n+1}}"'] & M^{+}_{\kappa^\flat}(\epsilon)/t^{(1-l)p^n} \arrow[u, "\wr", "f\mapsto f^{1/p^n}"']
\end{tikzcd}$$

where the bottom map is reduction mod $t^{(1-l)p^n}$. Since the two vertical maps are Hecke-equivariant by the first part, and the bottom map clearly is as well, so is the top map.

To deduce part 2, recall that the map (37) is semi-linear with respect to the map

$$\mathcal{O}_{K^\flat}/t^{(1-l)p^n} \to \mathcal{O}_{K^\flat}/t^{1-l} \to \mathcal{O}_K/p^{1-l}, \quad x \mapsto x^{1/p^n}.$$

This shows that in the inverse limit, the almost isomorphisms of part 3 fit together to give a Hecke-equivariant $\mathcal{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p^{1-l}$-linear almost isomorphism

$$M^+_{\kappa^\flat}(\epsilon) = \varprojlim_n M^+_{\kappa^\flat}(\epsilon)/t^{(1-l)p^n} \xrightarrow[\sim]{(f\mapsto f^{1/p^n})_{n\in\mathbb{N}}} \varprojlim_{f\mapsto f^p} M^+_{\kappa_n,\Gamma_0(p^n)}(\epsilon)/p^{1-l}.$$

This description also shows that we can recover the almost isomorphism in 3 from this as the projection from the inverse system. This finishes the proof of the theorem. □

The following commutative algebra lemmas were used in the proof:

Lemma 12.2.5. *Let K be a non-archimedean field extension of $\mathbb{Q}_p$ with non-discrete value group $|K| \subseteq \mathbb{R}$. Let $\delta \in \mathbb{R}_{>0}$ and let $\delta_0 \leq \delta_1 \leq \dots$ be a sequence in $|K|$ converging to δ. Recall that we write $p^\delta := \{x \in K \mid |x| \leq |p|^\delta\}$. Let M be any flat $\mathcal{O}_K$-module. Then the natural map $M/p^\delta \to \varprojlim_n M/p^{\delta_n}$ is an almost isomorphism.*

Proof. Since M is flat, applying $M \otimes_{\mathcal{O}_K} -$ to the short exact sequences of $\mathcal{O}_K$-modules $0 \to p^{\delta_n}\mathcal{O}_K/p^\delta \to \mathcal{O}_K/p^\delta \to \mathcal{O}_K/p^{\delta_n} \to 0$ gives an inverse system of short exact sequences

$$0 \to p^{\delta-\delta_n}M/p^\delta \to M/p^\delta \to M/p^{\delta_n} \to 0.$$

Let $C_n := p^{\delta_n}M/p^\delta$. Then for any $x \in \mathcal{O}_K$ with $|x| < 1$, multiplication by x on C_n is zero for $n \gg 0$ since $\delta_n \to \delta$. In other words, the pro-$\mathcal{O}_K$-module $(C_n)_{n\geq 1}$ is almost-pro-zero in the sense of [5], Definition 3.2. This means that $(C_n)_{n\geq 1}$ is x-torsion as a pro-$\mathcal{O}_K$-module, and in particular the same is true for $\varprojlim_N C_n$ and $R^1\varprojlim_n C_n$. The result therefore follows by applying $\varprojlim_n$ to the above system of short exact sequences. □

Lemma 12.2.6. *Let $0 \leq l < 1$. Let $h : M_1 \hookrightarrow M_2$ be a monomorphism of flat $(\mathcal{O}_K/p^{1-l})^a$-modules such that* coker h *is p^ϵ-torsion for some $0 \leq \epsilon < 1-l$. Then h is an isomorphism.*

Proof. By slightly increasing ϵ if necessary, we may without loss of generality assume that $\log|K|$. Write $C = \operatorname{coker} h$. Since M_1 and M_2 are flat, tensoring $h : M_1 \hookrightarrow M_2$ with the short exact sequence $\mathcal{O}_K/p^{1-\epsilon-l} \xrightarrow{\cdot p^\epsilon} \mathcal{O}_K/p^{1-l} \to \mathcal{O}_K/p^\epsilon$ gives a diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_1/p^{1-\epsilon-l} \arrow[r, "p^\epsilon"] \arrow[d, hook] & M_1 \arrow[r] \arrow[d, hook, "h"] & M_1/p^\epsilon \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M_2/p^{1-\epsilon-l} \arrow[r, "p^\epsilon"] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_2/p^\epsilon \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C/p^{1-\epsilon-l} \arrow[r, "p^\epsilon"] & C \arrow[r] & C/p^\epsilon \arrow[r] & 0
\end{tikzcd}$$

where the map $M_1/p^{1-l-\epsilon} \hookrightarrow M_2/p^{1-l-\epsilon}$ on the top vertical left is injective because the map h in the top vertical middle is. Similarly, $M_1/p^\epsilon \hookrightarrow M_2/p^\epsilon$ is injective by the same argument applied to the sequence $\mathcal{O}_K/p^\epsilon \xrightarrow{\cdot p^{1-l-\epsilon}} \mathcal{O}_K/p^{1-l} \to \mathcal{O}_K/p^{1-l-\epsilon}$. For this we may without generality assume that $1-l-\epsilon \in \log|K|$ by slightly increasing l if necessary. Injectivity of the map on the bottom left then follows by the Snake Lemma.

Since C is annihilated by p^ϵ, the map $C \to C/p^\epsilon C = C$ is an isomorphism, which implies $C/p^{1-\epsilon} = 0$. This means that multiplication by $p^{1-\epsilon}$ is surjective on C. But since $\epsilon < 1$, iterating $\cdot p^{1-\epsilon}$ shows that so is multiplication by p, which is already zero. Thus $C = 0$. □

13 Consequences for classical modular forms

Throughout, we fix a perfectoid field extension K of $\mathbb{Q}_p^{\mathrm{cyc}}$.

13.1 A "perfectoid" property of true p-adic modular forms

The tilting isomorphism has the following amusing "perfectoid" consequence: Any p-adic modular form admits arbitrary p-th roots mod p. More precisely:

Corollary 13.1.1. *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be a non-trivial p-adic weight. Let $f \in M_\kappa^+(\epsilon)$ be a p-adic modular form of weight κ. Then for any sequence of weights $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ with $\kappa_0 = \kappa$ and $\kappa_{n+1}^p \equiv \kappa_n \bmod p$ as in Theorem 12.2.1, there exists a sequence of p-adic modular forms $f_n \in M_{\kappa_n}^+(p^{-n}\epsilon)$ with $f_1 = f$ such that $f_{n+1}^{(p)} \equiv f_n \bmod p^{1-l}$.*

Proof. By Theorem 12.2.1.3, there is $f^\flat \in M_{\kappa^\flat}^+(\epsilon)$ such that $f^\flat \equiv f \bmod p^{1-l}$. By Theorem 12.2.1.2, this corresponds to a sequence $(f_n)_{n\in\mathbb{Z}_{\geq 0}}$ with $f_n \in M_{\kappa_n}^+(p^{-n}\epsilon)$ as desired. □

As we shall discuss next, one can upgrade Proposition 5.4.3 to obtain an example for such a system of roots in the case of Eisenstein series E_κ at classical weights:

Definition 13.1.2. Recall that a weight $\kappa : \mathbb{Z}_p^\times \to R^\times$ valued in a $\mathbb{Z}_p$-algebra R is called classical if there are $k \geq 2$ and a finite character $\mathbb{Z}_p^\times \to R^\times$ such that $\kappa(x) = \chi(x)x^k$. We then write $\kappa = (\chi, k)$. We denote by $M_\kappa^{\mathrm{cl}}(R) = \Gamma(X_R, \omega^\kappa)$ the R-module of classical modular forms of weight κ over R. For $R = K$, we have $M_\kappa^{\mathrm{cl}}(K) \subseteq M_\kappa(\epsilon)$ for any $0 \leq \epsilon < \epsilon_\kappa$.

We first convince ourselves that even though our notion of integrality in general depends on ϵ, for classical modular forms it ties in with the usual notion of integral forms as expected:

Lemma 13.1.3. *Let $\kappa = (\chi, k)$ be a classical weight. Then for any $0 \leq \epsilon < \epsilon_\kappa$,*

$$M_\kappa^{\mathrm{cl}}(K) \cap M_\kappa^+(\epsilon) = M_\kappa^{\mathrm{cl}}(\mathcal{O}_K).$$

In particular, we have $M_\kappa^{\mathrm{cl}}(\mathcal{O}_K)/p \hookrightarrow M_\kappa^+(\epsilon)/p$ for any such ϵ.

Proof. Let $f \in M_\kappa^{\mathrm{cl}}(\mathcal{O}_K)$. Then f is a section of ω^κ on $X_{\mathcal{O}_K}$, which is the same as a section of the completion $\mathfrak{w}^\kappa$ on $\mathfrak{X}_{\mathcal{O}_K}$ since $X_{\mathcal{O}_K}$ is proper. Using the commutative diagram

$$\begin{array}{ccccc} \mathcal{X}(\epsilon) & \longrightarrow & \mathcal{X} & \longrightarrow & X_K \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X}(\epsilon) & \longrightarrow & \mathfrak{X} & \longrightarrow & X_{\mathcal{O}_K} \end{array}$$

we see that the image of f in $M_\kappa(\epsilon)$ is already contained in the sections of $\mathfrak{w}^\kappa$ over $\mathfrak{X}(\epsilon)$. Thus f is already in $M_\kappa^+(\epsilon)$. This gives the inclusion from right to left.

To see the inclusion from left to right, let $f \in M_\kappa^{\mathrm{cl}}(K)$ and assume that the image of f in $M_\kappa(\epsilon)$ is bounded by 1 on $\mathcal{X}(\epsilon)$. Then it is in particular bounded by 1 on $\mathcal{X}(0)$, and thus f has integral q-expansion in $\mathcal{O}_K$. By the classical q-expansion principle, [34], Corollary 1.6.2, this shows that $f \in M_\kappa^{\mathrm{cl}}(\mathcal{O}_K)$. This gives the inclusion in the other direction. □

We can use the Lemma to upgrade the example of the Eisenstein series, Proposition 5.4.3, to an overconvergent version:

Corollary 13.1.4. *Let $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of classical weights satisfying $\kappa_{n+1}^p \equiv \kappa_n$ with $|T_{\kappa_0}| \geq |p|$ and $|T_{\kappa_1}| > |p|^{1/(p-1)}$. Let $\kappa^\flat$ be the corresponding t-adic weight.*

1. *The map $M_{\kappa_{n+1}}^+(p^{-n+1}\epsilon)/p^{1-l} \xrightarrow{a} M_{\kappa_n}^+(p^{-n}\epsilon)/p^{1-l}$ of Theorem 12.2.1.1 sends*

$$E_{\kappa_{n+1}} \mapsto E_{\kappa_n}.$$

2. *Under the isomorphism $M^+_{\kappa^\flat}(\epsilon) \overset{a}{=} \varprojlim_{f\mapsto f^{(p)}} M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}$ of Theorem 12.2.1.3, the t-adic modular form $E_{\kappa^\flat}$ of Lemma 5.4.2 corresponds to the sequence $(E_{\kappa_n})_{n\in\mathbb{Z}_{\geq 0}}$*

Here for both statements we use almost elements in the sense of [48], Proposition 4.6.

Proof. For the first part, we recall that the Atkin–Lehner isomorphism $M^+_{\kappa_{n+1}}(p^{-n-1}\epsilon) \xrightarrow{\sim} M^+_{\kappa_{n+1},\Gamma_0(p)}(p^{-n}\epsilon)$ sends $E_{\kappa_{n+1}} = E_{\kappa_{n+1}}(q)$ to a classical modular form with q-expansion $E_{\kappa_n}(q^{1/p}) \in \mathcal{O}_K[[q^{1/p}]]$. It is then clear that $E^p_{\kappa_{n+1}} \in M^{\mathrm{cl}}_{\kappa^p_{n+1}}(\mathcal{O}_K)$ and it suffices to prove that the isomorphism $M^+_{\kappa^p_{n+1}}(p^{-n}\epsilon)/p^{1-l} = M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}$ sends $E^p_{\kappa_{n+1}}$ to E_{κ_n}. By Lemma 13.1.3, this can be checked on q-expansions, where it follows from Proposition 5.4.3.

To see the second part, we note that the tilting isomorphism of Theorem 12.2.1 is compatible with the restriction maps in ϵ on both sides. We thus have a commutative diagram

$$\begin{tikzcd} M^+_{\kappa^\flat}(0) \arrow[r, "\sim"] & \varprojlim_{n\in\mathbb{N}} M^+_{\kappa_n}(0)/p^{1-l} \\ M^+_{\kappa^\flat}(\epsilon) \arrow[u, hook] \arrow[r, "\sim"] & \varprojlim_{n\in\mathbb{N}} M^+_{\kappa_{n+1}}(p^{-n}\epsilon)/p^{1-l}. \arrow[u] \end{tikzcd}$$

By the first part, the system $(E_{\kappa_n})_{n\in\mathbb{N}}$ defines an element on the bottom right. By Proposition 5.4.3, its image in the top right coincides with the image of $E_{\kappa^\flat}$ in the top left. By Lemma 5.4.2, $E_{\kappa^\flat}$ comes already from the bottom left. Since the vertical map on the left is injective, this shows that $(E_{\kappa_n})_{n\in\mathbb{Z}_{\geq 0}}$ is the image of $E_{\kappa^\flat}$ in the bottom left. □

13.2 Outlook and conjectures

We end our discussion of the tilting isomorphism of modular forms by returning to the initial question we started with, namely the geometry of the eigencurve. Recall that the motivation for the search for a theory of t-adic modular forms is the hope that the Hecke action on this module would help understand where the patterns predicted by Coleman's Spectral Halo Conjecture and Buzzard–Kilford's Conjecture come from. To this end, we shall briefly discuss some ideas on what role the tilting isomorphism could play in this, and give some precise conjectures.

As a first step, it seems natural to ask whether the tilting isomorphism of modular forms can be restricted to give a tilting isomorphism of eigenforms. More precisely, the example of Hida families and the Eisenstein family from Proposition 5.4.3 and Corollary 13.1.4 suggests the following incarnation of the Halo Conjecture:

Conjecture 13.2.1. *There is $1 > r > |p|$ for which the following holds: Let $(\kappa_n)_{n\in\mathbb{Z}_{\geq 0}}$, $\kappa^\flat$, ϵ and l be like in Theorem 12.2.1. Let m be such that $|T_{\kappa_m}| \geq r$. Then t-adic eigenforms $f^\flat \in M^+_{\kappa^\flat}(\epsilon)$ are in one-to-one correspondence with sequences of eigenforms $(f_n)_{n\geq m}$ with $f_n \in M^+_{\kappa_n}(p^{-n}\epsilon)$ such that $f^{(p)}_{n+1} \equiv f_n \bmod p^{1-l}$ up to equivalence, where two sequences are considered to be equivalent if they have the same image in $\varprojlim_{f\mapsto f^{(p)}} M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}$.*

Similarly, one could ask whether Corollary 13.1.1 holds for eigenforms:

Conjecture 13.2.2. *For $(\kappa_n)_{n\in\mathbb{Z}_{\geq 0}}$, ϵ and l as above and any eigenform $f_m \in M^+_{\kappa_m}(p^{-m}\epsilon)$, there is a sequence of eigenforms $f_n \in M^+_{\kappa_n}(p^{-n}\epsilon)$ such that $f^{(p)}_{n+1} \equiv f_n \bmod p^{1-l}$.*

Remark 13.2.3. The first conjecture is essentially a question about lifting to p-adic eigenforms: It is clear from the Hecke-equivariance of the tilting isomorphism in Theorem 12.2.1.2 that any sequence of eigenforms $(f_n)_{n\geq m}$ as in the conjecture gives rise to a t-adic overconvergent eigenform $f^\flat$. Conversely, by Theorem 12.2.1.3 any t-adic overconvergent eigenform $f^\flat$ gives rise to a system of Hecke eigenvectors in $M^+_{\kappa_n}(p^{-n}\epsilon)/p^{1-l}$. The question is thus whether any such eigenvector can be lifted to an eigenform in $M^+_{\kappa_n}(p^{-n}\epsilon)$ for $|T_{\kappa_n}| \geq r$.

We note that this is different to the classical Deligne–Serre lifting situation that deals with finite Hecke algebras over discrete valuation rings, since we are working over non-Noetherian $\mathcal{O}_K$-algebras, and reduce modulo the ideal p^{1-l} rather than modulo the maximal ideal.

If the first conjecture is true, then the second conjecture is, on the other hand, a question about lifting to t-adic eigenforms: The eigenform f_m reduces to a Hecke-eigenvector $\bar{f}_m$ of $M^+_{\kappa_m}(p^{-m}\epsilon)/p^{1-l} = M^+_{\kappa^\flat}(\epsilon)/t^{(1-l)p^m}$ by Theorem 12.2.1.3. If Conjecture 13.2.1 is true, then any lift to $M^+_{\kappa^\flat}(\epsilon)$ gives rise to a system of eigenforms $(f_n)_{n\geq m}$ verifying Conjecture 13.2.2.

Example 13.2.4. The reader is invited to experiment with Conjecture 13.2.2 by computing spaces of classical modular forms. Here is a concrete example: Let $p = 2$, fix an embedding $\mathbb{Z}_2^{\mathrm{cyc}} \hookrightarrow \mathbb{C}_2$ and a compatible system $(\zeta_{2^n})_{n\in\mathbb{N}}$ of primitive 2^n-th unit roots. For $n \in \mathbb{Z}_{\geq 0}$, let $\chi_n : (\mathbb{Z}/2^{n+2}\mathbb{Z})^\times \to \mathbb{Z}[\zeta_{2^{n+1}}]^\times$ be the Dirichlet character defined by $-1 \mapsto 1$ and $5 \mapsto \zeta_{2^{n+1}}$. Consider the classical weights $\kappa_n = (\chi_n, 2)$, then $\kappa_{n+1}^2 \equiv \kappa_n \bmod 2$ for all n and $|T_{\kappa_n}| = 2^{-1/2^n}$. Then under the canonical isomorphism $\mathbb{Z}_2^{\mathrm{cyc}\flat} = \mathbb{F}_2[[t^{1/2^\infty}]]$ sending $(\zeta_{2^{n+1}} - 1)_{n\in\mathbb{Z}_{\geq 0}} \leftarrow\!\shortmid t$, the sequence $(\kappa_n)_{n\in\mathbb{Z}_{\geq 0}}$ corresponds to the canonical t-adic weight $\kappa^\flat = \overline{\kappa} : \mathbb{Z}_2^\times \to \mathbb{F}_2[[t^{1/2^\infty}]]^\times$ sending $-1 \mapsto 1$ and $5 \mapsto 1 + t$.

The space of classical modular forms $M^{\mathrm{cl}}_{\kappa_0}(\mathbb{Q})$ has dimension 2, and the two normalised eigenforms have U_2-eigenvalues 1 and 2. The first of these is the Eisenstein series

$$E_{\kappa_0} = 1 - 2q - 2q^2 + 4q^3 - 2q^4 + 8q^5 + 4q^6 + \dots$$

for which we have already verified Conjecture 13.2.2 in Corollary 13.1.4. Let f_{κ_0} be the other normalised eigenform in $M^{\mathrm{cl}}_{\kappa_0}(\mathbb{Q})$, with U_p-eigenvalue 2. Then using sage [55], we compute that there are eigenforms $f_{\kappa_1} \in M^{\mathrm{cl}}_{\kappa_2}(\mathbb{Q}(\zeta_4))$ and $f_{\kappa_2} \in M^{\mathrm{cl}}_{\kappa_3}(\mathbb{Q}(\zeta_8))$ with q-expansions

$$\begin{aligned}
f_{\kappa_0} &= q+2q^2+ 2q^3+4q^4+ 4q^5+ 4q^6 + \dots\\
f_{\kappa_1} &= q+2q^2+(-\zeta_4 + 3)q^3+4q^4+(\zeta_4 + 5)q^5+(-2\zeta_4 + 6)q^6 + \dots\\
f_{\kappa_2} &= q+2q^2+ (\zeta_8^3 + 3)q^3+4q^4+(\zeta_8 + 5)q^5+ (2\zeta_8^3 + 6)q^6 + \dots
\end{aligned}$$

One verifies on the level of q-expansion coefficients that these satisfy $f^{(2)}_{\kappa_2} \equiv f_{\kappa_1} \bmod 2$ and $f^{(2)}_{\kappa_1} \equiv f_{\kappa_0} \bmod 2$ inside $\mathbb{Z}_p^{\mathrm{cyc}}$. Conceptually, the reason for this is that there is a second kind of Eisenstein family (see [14], §1) which for a classical weight $\kappa = (\chi, k)$ is of the form

$$f_\kappa = \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1}\chi(m/d) \right) q^m.$$

In particular, like in Proposition 5.4.3, one sees that the f_{κ_n} satisfy $f^{(2)}_{\kappa_{n+1}} \equiv f_{\kappa_n} \bmod 2$ and thus converge to a t-adic Eisenstein series $f_{\kappa^\flat}$ with q-expansion

$$f_{\kappa^\flat} = \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \left(\sum_{d|m} d^{k-1}\kappa^\flat(m/d) \right) q^m \in \mathbb{F}_2[[t]][[q]].$$

Here for even m the coefficients of $f_{\kappa^\flat}$ vanish because all even coefficients of all f_{κ_n} are divisible by 2. In particular, $f_{\kappa^\flat}$ satisfies $U_2(f_{\kappa^\flat}) = 0$ and is thus of infinite slope.

One obtains further examples if one lets non-Eisenstein eigenforms enter the picture: The space $M^{\mathrm{cl}}_{\kappa_1}(\mathbb{Q}(\zeta_4))$ has dimension 3, the remaining normalised cuspidal eigenform being

$$g_{\kappa_1} = q - (\zeta_4 + 1)q^2 + (\zeta_4 - 1)q^3 + 2\zeta_4 q^4 - (\zeta_4 + 1)q^5 + 2q^6 + \dots$$

and this also satisfies $g^{(2)}_{\kappa_1} \equiv f_{\kappa_0} \bmod 2$, but has U_2-eigenvalue $-(\zeta_4 + 1)$ and thus slope $\frac{1}{2}$, in contrast to f_{κ_1} which had eigenvalue 2 and slope 1. The space $M^{\mathrm{cl}}_{\kappa_2}(\mathbb{Q}(\zeta_8))$ is of dimension 5, and apart from the two Eisenstein series, it contains the cuspidal eigenform

$$h_{\kappa_2} = q - (\zeta_8^3 + \zeta_8)q^2 + (\zeta_8^2 + \zeta_8)q^3 - 2q^4 + (2\zeta_8^3 - 2\zeta_8^2 - \zeta_8 - 1)q^5 + (-\zeta_8^3 - \zeta_8^2 + \zeta_8 + 1)q^6 + \dots$$

which has slope $\frac{1}{2}$ and satisfies $h_{\kappa_2}^{(2)} \equiv f_{\kappa_1}$. The remaining two eigenforms in $M_{\kappa_2}^{\mathrm{cl}}(\mathbb{Q}(\zeta_8))$ are a pair of Galois conjugates $g_{\kappa_2}, g'_{\kappa_2}$ defined over a quadratic extension L of $\mathbb{Q}(\zeta_8)$ in which 2 is split. They satisfy $g_{\kappa_2}^{(2)} \equiv g_{\kappa_1}$ or $g_{\kappa_2}^{\prime(2)} \equiv g_{\kappa_1}$ mod 2 in $\mathbb{C}_p$, each equation holding for precisely one of the two choices of $\mathbb{Q}(\zeta_8)$-linear embedding $L \hookrightarrow \mathbb{C}_2$. Let us fix the embedding for which the first congruence holds, then g_{κ_2} has slope $\frac{1}{4}$ and g'_{κ_2} has slope $\frac{3}{4}$.

The point is that under the relation in Conjectures 13.2.1 and 13.2.2, the different sequences that end in $\dots, f_{\kappa_2}, f_{\kappa_1}, f_{\kappa_0}$ and $\dots, h_{\kappa_2}, f_{\kappa_1}, f_{\kappa_0}$ and $\dots, g_{\kappa_2}, g_{\kappa_1}, f_{\kappa_0}$ would correspond to different eigenforms of $M_{\kappa^\flat}^{+}(\epsilon)$ of slopes ≥ 3 and $= 2$ and $= 1$, respectively.

Remark 13.2.5. The relation to the conjectures of Coleman and Buzzard–Kilford is as follows: Let us first observe that if Conjectures 13.2.1 and 13.2.2 are true, then via the map

$$M_{\kappa^\flat}^{+}(\epsilon)/t^{(1-l)p^n} \to M_{\kappa_n}^{+}(p^{-n}\epsilon)/p^{1-l}, \quad f \mapsto f^{(1/p^n)}$$

from Theorem 12.2.1.3, and by letting $\epsilon \to 0$, $l \to 0$, one can identify the set of p-adic slopes of $M_{\kappa_n}^{+}(p^{-n}\epsilon)$ in the half-open interval $[0,1)$ with the t-adic slopes of $M_{\kappa^\flat}^{+}(\epsilon)$ in the interval $[0,p^n)$ via the rescaling of multiplication by p^n, since this is the effect that raising to the p^n-th power has on valuations. This is interesting for several reasons: First, it immediately shows that this set of slopes only depends on $|T_{\kappa_n}|$, as predicted by the Spectral Halo conjecture.

Second, and similarly, via the canonical "Frobenius" map from Theorem 12.2.1.1,

$$M_{\kappa_{n+1}}^{+}(p^{-(n+1)}\epsilon)/p^{1-l} = M_{\kappa^\flat}^{+}(\epsilon)/t^{(1-l)p^{n+1}} \to M_{\kappa^\flat}^{+}(\epsilon)/t^{(1-l)p^n} = M_{\kappa_n}^{+}(p^{-n}\epsilon)/p^{1-l}, f \mapsto f^{(p)},$$

this would imply that the slopes of $M_{\kappa_{n+1}}^{\dagger} := \varprojlim_{0<\epsilon\to 0} M_{\kappa_{n+1}}(\epsilon)$ in the interval $[0,1/p)$ are in bijection with the slopes of $M_{\kappa_n}^{\dagger}$ in the interval $[0,1)$ via multiplication by p.

We note that at a classical point (χ,k), all eigenforms with slope in $[0,1)$ are classical by Coleman's Control Theorem. Together with the Atkin–Lehner symmetry in the classical slopes used in [4] and [42], Proposition 3.22, this also determines the slopes in $(k-2,k-1]$.

These phenomena combined already explain, at least partially, the arithmetic progressions in the slopes of $M_{\kappa^\flat}^{+}$ predicted by the conjecture of Buzzard–Kilford. As an illustrative example, let us revisit Example 13.2.4 of $p=2$ and consider the eigenform $g_{\kappa_1} \in M_{\kappa_1}^{\mathrm{cl}}(\mathbb{Q}(\zeta_4))$: We read from its q-expansion that this has U_2-eigenvalue $-(\zeta_4+1)$ and thus slope $\frac{1}{2}$. The other slopes in $M_{\kappa_1}^{\mathrm{cl}}(\mathbb{Q}(\zeta_4))$, from two Eisenstein series, are 0 and 1.

Assume now that Conjecture 13.2.2 is true in the case of $N=1$ (throughout we assume $N \geq 3$, but for this Example there is no harm in taking $N=1$. One may define the relevant spaces of modular forms for $N=1$ using diamond operators). Let then g_{κ_n} for $n \geq 3$ be a sequence of normalised eigenforms of weights $\kappa_n = (\chi_n, 2)$ such that $g_{\kappa_{n+1}}^{(p)} = g_{\kappa_n}$ for all $n \geq 1$. Then by comparing coefficients of q^2, we see that g_n will have slope $\frac{1}{2^{n-1}}$.

By the Atkin–Lehner symmetry, we see that since g_{κ_2} has slope $\frac{1}{4}$, there is in $M_{\kappa_2}^{\mathrm{cl}}(\mathbb{C}_p)$ also an eigenform of slope $1-\frac{1}{4}=\frac{3}{4}$ (namely g'_{κ_2} in Example 13.2.4). By the rescaling observation from the beginning of this Remark, this means that there is in $M_{\kappa_2}^{\mathrm{cl}}(\mathbb{Q}(\zeta_4))$ an eigenform of slope $\frac{1}{2}\cdot\frac{3}{4}=\frac{3}{8}$. Iterating this argument, and combining it with well-known formulas for the dimension of $M_{\kappa_n}^{\mathrm{cl}}(\mathbb{C}_p)$ in order to determine the number of eigenforms of slope $\frac{1}{2}$ in $M_{\kappa_n}^{\mathrm{cl}}(\mathbb{C}_p)$, this shows inductively that the slopes of $M_{\kappa_n}^{\mathrm{cl}}(\mathbb{C}_p)$ are given by

$$0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \dots, \frac{2^n-1}{2^n}, 1,$$

thus recovering Corollary (i) to Theorem B of [14]. Since by the observation in the beginning of this Remark, the slopes of $M_{\kappa^\flat}^{\dagger}$ in the interval $[0,2^n)$ are precisely the slopes of $M_{\kappa_n}^{\mathrm{cl}}(\mathbb{C}_p)$ multiplied by 2^n, this shows that the slopes of U_2 acting on $M_{\kappa^\flat}^{\dagger}$ are given by

$$0, 1, 2, 3, \dots, \infty$$

in accordance with Theorem B of [14].

Going into a slightly different direction, the tilting isomorphism might be useful to answer the following related question about the geometry of the eigencurve at the boundary:

Question 13.2.6. Is the adic eigencurve smooth over the boundary?

While the results of [14] imply that this is true for $p = 2$ and $N = 1$, this question does not seem to be covered by the Spectral Halo Conjecture, or recent advances like [42] or [4].

Remark 13.2.7. A strategy to attack this question might be to feed the fact that the eigencurve is étale at classical points into the tilting isomorphism: Since Conjecture 13.2.1 would imply that t-adic eigenform of finite slope can be described as compatible sequences of classical p-adic eigenforms, a natural idea would be to use the tilting equivalence of étale sites, [48], Theorem 7.12, to translate the étaleness into characteristic p.

Remark 13.2.8. Either of the potential applications of the tilting isomorphisms sketched in the above Remarks would require a good understanding of how the Hecke algebras can be traced through the tilting equivalence. But already determining the correct integral structure of the Hecke algebra is a subtle question: Recall that so far we have only discussed the "integral Hecke algebra", namely the $\mathbb{Z}_p$-span of the Hecke operators.

If $\mathbb{T}_\kappa$ is the Hecke algebra acting on a classical space $M^{\mathrm{cl}}_\kappa(K)$, then apart from this "integral Hecke algebra" $\mathbb{T}^{\mathrm{int}}_\kappa$ there are two weaker natural notions of what it could mean for an operator to be integral: The first is to consider the subspace $\mathbb{T}^+_\kappa$ of elements that are integral with respect to the operator norm, namely fix the subspace $M^{\mathrm{cl}}_\kappa(\mathcal{O}_K)$. This is the notion of integrality that is easiest to work with in the context of the tilting isomorphism, since like $\mathbb{T}^{\mathrm{int}}_\kappa$, it induces a Hecke action on $M^{\mathrm{cl}}_\kappa(\mathcal{O}_K)/p$.

On the other hand, one may consider the integral subspace with respect to the spectral norm: For this one can use the natural K-Banach algebra structure of $\mathbb{T}_\kappa$ and consider the subspace of power-bounded elements $\mathbb{T}^\circ_\kappa$. Equivalently, this is the space of operators which fix the $\mathcal{O}_K$-lattice of normalised eigenforms. This notion of integrality is clearly better suited to study for example étaleness, since $\mathbb{T}^\circ_\kappa$ is almost étale if $\mathbb{T}_\kappa$ is étale.

While one always has $\mathbb{T}^+_\kappa \subseteq \mathbb{T}^\circ_\kappa$ (by "Newton above Hodge" for linear operators), these different integral subrings do not in general agree as the following example shows:

Let $p = 2$, let $\chi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ be the Dirichlet character of conductor 8 determined by $5 \mapsto -1$ and $7 \mapsto -1$ and let $\kappa = (\chi, 3)$. Then the classical space of modular forms $M^{\mathrm{cl}}_\kappa(\mathbb{Q})$ has dimension 3, and one can use sage [55] in conjunction with linear algebra tools like Smith Normal Forms to compute

$$\mathbb{T}^{\mathrm{int}}_\kappa = \langle T_1, T_2, T_3 \rangle, \quad \mathbb{T}^{\mathrm{cl},+}_\kappa = \Big\langle T_1, T_2, \frac{1}{2}T_3 \Big\rangle, \quad \mathbb{T}^\circ_\kappa = \Big\langle T_1, T_2, \frac{1}{2}T_1 + \frac{1}{4}T_3, \Big\rangle.$$

A Lemmas in topological algebra

In this appendix, we prove various Lemmas for topological algebras. The first two subsections can be roughly summarised as discussing integral closure properties in profinite and perfectoid covers. The last subsection discusses various aspects of complexes of adic rings, and in particular several situations in which exactness is preserved by adic completions.

A.1 Integral closures from perfectoid covers

Let K be a perfectoid field of arbitrary characteristic with ring of integers $\mathcal{O}_K$ and pseudo-uniformiser π. In this subsection, we give the following method to prove that an $\mathcal{O}_K$-algebra of topologically finite presentation is integrally closed in its generic fibre:

Proposition 4.1.4. *Let K be a perfectoid field with pseudo-uniformiser $\pi \in \mathcal{O}_K$. Let A be a flat $\mathcal{O}_K$-algebra of topologically finite presentation. Assume that there is a map $h : A \to A_\infty$ into a flat $\mathcal{O}_K$-algebra A_∞ such that A_∞^a is a perfectoid $\mathcal{O}_K^a$-algebra and such that the reduction $h : A/\pi \to A_\infty/\pi$ is almost injective. Then A is integrally closed in $A[1/\pi]$.*

For the proof we need the following three algebra lemmas, which do not require perfectoid assumptions: Instead, let us fix an f-adic valuation ring R. We assume that R is non-discrete, so that we may work in the almost setting with respect to the maximal ideal $\mathfrak{m}$ of R.

Lemma A.1.1.

1. *Suppose that S is an R-algebra of topologically finite presentation which is almost zero, i.e. S is annihilated by $\mathfrak{m}$. Then $S = 0$.*
2. *Let A be an R-algebra of topologically finite presentation and let M be an A-module of finite presentation. Assume that M is almost zero. Then $M = 0$.*

As the residue field $k := R/\mathfrak{m}$ shows, it does not suffice to assume that S is of finite type.

Proof. Write $S = R\langle X_1, \dots, X_n\rangle/I$ where I is a finitely generated ideal. Since S is annihilated by $\mathfrak{m}$, we have $\mathfrak{m} \subseteq I$. In particular, $S = R[X_1, \dots, X_n]/I$ is algebraically finitely presented and we can write $S = k[X_1, \dots, X_n]/I$, with R-algebra structure via $R \to k$.

Suppose towards a contradiction that $1 \notin I$. Then I is contained in a maximal ideal $\mathfrak{n} \subseteq k[X_1, \dots, X_n]$ and we have a projection $S \twoheadrightarrow S/\mathfrak{n} = k'$ where k' is some finite extension of k. Since $\mathfrak{n}$ is finitely generated, and S is finitely presented over R, the same is true for k'. By [24], Proposition 1.4.7, this implies that k' is finitely presented over R as a module. Choosing any k-module splitting $k' \twoheadrightarrow k$, this implies that also k is finitely presented as an R-module. But then the kernel $\mathfrak{m}$ of the projection $R \to k$ has to be finitely generated. This contradicts the assumption that R is non-discrete. This proves the first part.

For part 2, we use that by [11], Corollary 7.3.6, the assumptions imply that the A-module M is already coherent, i.e. every finite submodule of M is of finite presentation. Let $m \in M$ and consider the submodule $M_0 := Am \subseteq M$. Since M_0 is finitely presented, the kernel of

$$0 \to I \to A \xrightarrow{a \mapsto am} M_0 \to 0$$

is a finitely generated ideal of A. Thus A/I is an R-algebra of topologically finite presentation that is almost zero as an R-module. By part 1, this implies $A/I = 0$ and thus $m = 0$. □

Lemma A.1.2. *Let $f : A \to B$ be an integral morphism of R-algebras. Assume that A and B are flat, that A is of topologically finite presentation, and that f is an almost isomorphism. Then f is already an honest isomorphism.*

Proof. Since R is a valuation ring, an R-module is flat if and only if it is $\mathfrak{m}$-torsionfree.

Since A is flat, it is clear that f is injective: By assumption, $\ker f$ is a submodule of the $\mathfrak{m}$-torsionfree R-module A that is annihilated by $\mathfrak{m}$, which implies $\ker f = 0$.

It therefore suffices to prove that $\operatorname{coker} f = 0$. For this it suffices to consider finite subextensions $f : A \to A[x] \subseteq B$ for any $x \in B$, which are again flat over R since they are submodules of the $\mathfrak{m}$-torsionfree R-algebra B. We may therefore assume that f is finite.

By a theorem of Raynaud–Gruson [11], Theorem 7.3.4, the assumptions that $A \to B$ is finite and B is flat over R imply that B is of topologically finite presentation over R, and thus also over A. Since B is finite over A, this implies that B is already of algebraically finite presentation over A. By [24], Proposition 1.4.7, the A-algebra B is then also of finite presentation as an A-module. Then also $\operatorname{coker}(f)$ is of finite presentation as an A-module. Since it is also almost zero by assumption, we have $\operatorname{coker}(f) = 0$ by Lemma A.1.1.2. □

proof of Proposition 4.1.4. Let A^{int} and A^{int}_∞ be the integral closures of A in $A[1/\pi]$ and of A_∞ in $A_\infty[1/\pi]$, respectively. Consider the commutative diagram of short exact sequences

$$\begin{tikzcd} A_\infty \arrow[r, hook] & A^{\mathrm{int}}_\infty \arrow[r] & C_\infty \\ A \arrow[u] \arrow[r, hook] & A^{\mathrm{int}} \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

where the rightmost column is defined to be the cokernel of the horizontal maps. To prove the Proposition, we need to prove that $C = 0$. Since A is of topologically finite presentation, it suffices by Lemma A.1.2 to prove that C is almost zero.

To see this, we first note that since A_∞ is a flat $\mathcal{O}_K$-algebra, we have $A_\infty \subseteq A_\infty[1/\pi]^\circ$, and since the latter is integrally closed in particular $A^{\mathrm{int}}_\infty \subseteq A_\infty[1/\pi]^\circ$. On the other hand, since A^a_∞ is perfectoid, $A_\infty \subseteq A^{\mathrm{int}}_\infty \subseteq A_\infty[1/\pi]^\circ \overset{a}{=} A_\infty$, by [48], Theorem 5.2. Therefore $A_\infty \overset{a}{=} A^{\mathrm{int}}_\infty$ which shows that C_∞ is almost zero.

To deduce that the same is true for C, it suffices to see that $C \hookrightarrow C_\infty$ has almost zero kernel. Equivalently, it suffices to see that the diagram on the left is almost Cartesian (i.e. the map into the fibre product module is an almost isomorphism). But this follows from the fact that $A^{\mathrm{int}} \subseteq A[1/\pi]$ and that $A/\pi \hookrightarrow A_\infty/\pi$ is almost injective by assumption. □

A.2 Permanence properties of inverse limits of Huber rings

The following Lemma is often implicitly used in the context of perfectoid spaces, see for instance [7], proof of Lemma 3.20.

Lemma A.2.1. *Let A be a Tate ring with pseudo-uniformiser ϖ. If A has a ring of definition A_0 that is integrally closed in A, then $\varpi A^\circ \subseteq A_0$. In particular, A is uniform.*

Proof. By assumption, the ring A_0 is open, bounded and integrally closed in A. Let $x \in A^\circ$. Then x is power-bounded, so since A_0 is open there is $m \in \mathbb{N}$ such that $\varpi^m\{x^n|n \in \mathbb{N}\} \subseteq A_0$. But then $(\varpi x)^m \in A_0$. Since A_0 is integrally closed in A, this implies $\varpi x \in A_0$. This proves the first assertion. The second follows since $\varpi^{-1}A_0 \subseteq A$ is bounded and thus so is A°. □

The following is closely related to (but slightly different from) the discussion in §5 [6].

Lemma A.2.2. *Let R be an algebra and let $\pi \in R$ be a non-zero-divisor. Let $(A_i)_{i\in I}$ be a direct system of π-torsionfree R-algebras. Assume that each $A_i \subseteq A_i[1/\pi]$ is integrally closed. Let $A := (\varinjlim A_i)^\wedge$ be the π-adic completion. Then*

1. *The ring A is integrally closed in $A[1/\pi]$.*

2. *Assume moreover that all $(A_i[1/\pi], A_i)$ are Tate algebras with pseudo-uniformiser π. Then the Huber pair $(A[1/\pi], A)$ is uniform.*

3. *Assume moreover that $A_i/\pi \hookrightarrow A_j/\pi$ is injective for all $j \geq i$ (this is for example the case if each $A_i \to A_j$ is injective and integral), that the A_i are complete and that for all i we even have $A_i[1/\pi]^\circ = A_i$. Then $A[1/\pi]^\circ = A$.*

Proof. For the first part we let $B = \varinjlim A_i$. Then B is π-torsionfree and integrally closed in $B[1/\pi] = \varinjlim A_i[1/\pi]$: Given any $x \in B[1/\pi]$ that is integral over B, say $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$, there is i such that $x \in A_i[1/\pi]$, all $a_k \in A_i$, and the equation already holds in A_i. Then $x \in A_i$ since $A_i \subseteq A_i[1/\pi]$ is integrally closed, thus $x \in B$.

Since $A = \widehat{B}$, part 1 now follows from an approximation argument, see [6], Lemma 5.1.2. The second part follows immediately from the first part and Lemma A.2.1.

For part 3, we first explain the remark in parenthesis: If $A_i \to A_j$ is injective and finite, then $A_j \cap A_i[1/\pi] = A_i$ because any $x \in A_j \cap A_i[1/\pi]$ is finite over A_i, but $A_i \subseteq A_i[1/\pi]$ is integrally closed, thus $x \in A_i$. This shows that $A_i/\pi \hookrightarrow A_j/\pi$ is injective.

Next, we note that the condition that $A_i/\pi \hookrightarrow A_j/\pi$ be injective ensures that $A_i \hookrightarrow A$ is injective: Indeed, since π is a non-zero-divisor, an inductive argument using the short exact sequence $0 \to A_i/\pi^{n-1} \to A_i/\pi^n \to A_i/\pi \to 0$ shows that $A_i/\pi^n \hookrightarrow A_j/\pi^n$ is then also injective, and the statement follows in the inverse limit since A_i is π-adically separate.

It also implies that $A_i/\pi \hookrightarrow A/\pi$ is injective, and we thus have inside $A[1/\pi]$ that

$$A_i[1/\pi] \cap A = A_i. \tag{38}$$

Let now $x \in A[1/\pi]^\circ$. Then there is $n \in \mathbb{N}$ such that $\pi^n x \in A$. We can now find i and $x_i \in A_i$ which approximate $\pi^n x$ such that in A we have

$$\pi^n x = x_i + \pi^n a$$

for some $a \in A$. Then $x_i/\pi^n = x - a$. The right hand side of the latter is in $A[1/\pi]^\circ$, while the left hand side is in $A_i[1/\pi]$. It thus suffices to prove that

$$A[1/\pi]^\circ \cap A_i[1/\pi] = A_i[1/\pi]^\circ,$$

because by assumption $A_i[1/\pi]^\circ = A_i$, thus $x_i/\pi^n \in A_i$ and $x = x_i/\pi^n + a \in A$ as desired.

To show that the displayed equality holds, let $x \in A[1/\pi]^\circ \cap A_i[1/\pi]$. Then since $A \subseteq A[1/\pi]$ is open and $\{x^n | n \in \mathbb{N}\}$ is bounded in $A[1/\pi]$, there is $m \in \mathbb{N}$ such that $\pi^m x^n \in A$ for all n. It is clearly also in $A_i[1/\pi]$. By equation (38), we thus have $\pi^m x^n \in A_i[1/\pi] \cap A = A_i$ for all n. But this means that x is power-bounded in A_i, and thus $x \in A_i[1/\pi]^\circ = A_i$. □

Lemma A.2.2 combines with the following lemma to show that projective limits of normal admissible formal schemes have locally uniform generic fibre (we do not claim that the generic fibre is *stably* uniform since the formal topology is too coarse to see rational localisations).

Lemma A.2.3. *Let K be a non-archimedean field with pseudo-uniformiser π. Let A be a flat topologically finite type $\mathcal{O}_K$-algebra. Then $A[1/\pi]^\circ$ is the integral closure of A in $A[1/\pi]$.*

Proof. Any surjection $\alpha : \mathcal{O}_K\langle X_1, \dots, X_n\rangle \twoheadrightarrow A$ induces a projection of affinoid K-algebras

$$\alpha : K\langle X_1, \dots, X_n\rangle \twoheadrightarrow A[1/\pi]$$

which induces a residue norm $|\cdot|_\alpha$ on $A[1/\pi]$ by §3.1 [11]. By [11], Theorem 3.1.17, an element $x \in A[1/\pi]$ is power-bounded if and only if x is integral over the subring $A_0 := \{x \in A[1/\pi] \mid |x|_\alpha \leq 1\}$. But by definition we clearly have $A_0 = \alpha(\mathcal{O}_K\langle X_1, \dots, X_n\rangle) = A$. □

Under slightly stronger assumptions, Lourenço [43], §6 and Johansson–Newton [31], Appendix A, prove similar results for pseudorigid spaces: Let us for simplicity assume that $\mathfrak{X}$ is monogeneous for some element s, i.e. the analytic fibre is obtained by inverting s.

Lemma A.2.4 ([43], Lemma 6.1)**.** *Let $\mathfrak{X}$ be a normal excellent formal scheme, monogeneous with analytic fibre $\mathcal{X} = \mathfrak{X}^{\mathrm{ad}}_\eta$. Then the natural map $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \mathcal{O}^+_{\mathcal{X}}(\mathcal{X})$ is an isomorphism.*

Lemma A.2.5 ([31], Corollary A.6)**.** *Let $\mathfrak{X}$ be a normal Noetherian formal scheme formally of topologically finite type over a complete discrete valuation ring, monogenous with analytic fibre $\mathcal{X} = \mathfrak{X}^{\mathrm{ad}}_\eta$. Then the restriction map $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \mathcal{O}^+_{\mathcal{X}}(\mathcal{X})$ is an isomorphism.*

Proof. Since $\mathfrak{X}$ is normal, it is in particular reduced. By [31], Corollary A.6, it follows that for any affinoid $U \subseteq \mathcal{X}$, we have $\mathcal{O}^+_{\mathcal{X}}(U) = \mathcal{O}_{\mathcal{X}}(U)^\circ$. Since $\mathfrak{X}$ is normal, if follows from [20], Lemma 1.2.1 that for any affine open $U \subseteq \mathfrak{X}$, the algebra $\mathcal{O}_{\mathfrak{X}}(U)$ is normal. By the assumption that $\mathfrak{X}$ is monogeneous, we have $\mathcal{O}^+_{\mathcal{X}}(U) = \mathcal{O}_{\mathfrak{X}}(U)^{\mathrm{int}} = \mathcal{O}_{\mathfrak{X}}(U)$ where $\mathcal{O}_{\mathfrak{X}}(U)^{\mathrm{int}}$ is the integral closure in $\mathcal{O}_{\mathfrak{X}}(U)$. The general case follows by covering $\mathfrak{X}$ by affine opens. □

A.3 Complexes of topological algebras

A.3.1 Boundedness of torsion in complexes of Banach rings

Fix a non-archimedean field K with ring of integers $\mathcal{O}_K$ and pseudo-uniformiser ϖ.

Lemma A.3.1. *Let $C^\bullet$ be a complex over $\mathcal{O}_K$ such that each C^n is a flat Banach $\mathcal{O}_K$-module. Assume that C^n is open and bounded in $C^n[1/\varpi]$. Then for any $q \in \mathbb{Z}$, if the cohomology $H^q(C^\bullet[1/\varpi])$ vanishes, then there is $N > 0$ such that ϖ^N annihilates $H^q(C^\bullet[1/\varpi])$.*

Proof. See for instance [6] Proposition 9.3.3.2: Let us write d for the differentials in $C^\bullet$, then $d^{q-1}[1/\varpi] : C^{q-1}[1/\varpi] \to \ker d^q[1/\varpi]$ is surjective, and thus $d^{q-1}(C^{q-1}) \subseteq \ker d^q[1/\varpi]$ is open by the Open Mapping Theorem. Since $\ker d^q \subseteq \ker d^q[1/\varpi]$ is open and bounded, this implies that there is N such that $t^N \ker d^q \subseteq d^{q-1}(C^{q-1})$. □

Lemma A.3.2. *Let X be an affinoid rigid space over K. Let $\mathfrak{X}$ be a flat formal model of X over $\mathcal{O}_K$. Let F be a vector bundle on $\mathfrak{X}$. Let $\mathfrak{U}$ be a finite affine cover of $\mathfrak{X}$ with affine intersections. Then there is $N > 0$ such that t^N annihilates $\check{H}^q(\mathfrak{U}, F)$ for all $q > 0$.*

Proof. The Lemma follows from Lemma A.3.1 when we can show that the assumptions are satisfied by the Čech complex $\check{C}(\mathfrak{U}, F)$, which we will now check:

Let ϖ be a pseudo-uniformiser of K. By the assumption that F is a vector bundle on F, the $\mathcal{O}_X$-module $F[1/\varpi]$ is coherent. By Kiehl's Theorem B, [39], Satz 2.4, it is therefore acyclic on any affine open $U \subseteq X$, i.e. $H^q(U, F[1/\varpi]) = 0$ for $q > 0$. In particular, this applies to the generic fibre of any intersection V of opens in $\mathfrak{U}$. The Čech-to-sheaf spectral sequence therefore degenerates and shows that $H^q(\check{C}^\bullet(\mathfrak{U}, F))[1/\varpi] = \check{H}^q(\mathfrak{U}, F[1/\varpi]) = H^q(X, F[1/\varpi]) = 0$ for $p > 0$ by the assumption that X is itself affinoid.

For the Lemma to apply, we are therefore left to see that for any affine $U \subseteq \mathfrak{X}$, the $\mathcal{O}_K$-module $F(U)$ is a flat Banach $\mathcal{O}_K$-module that is open and bounded in $F(U)[1/\varpi]$. To this end, choose any affine cover $U = \cup U_i$ that trivialises F, then $F(U)$ is the kernel of the map $\delta : \prod F(U_i) \to \prod F(U_{ij})$ which is isomorphic to a linear map $\prod \mathcal{O}_{\mathfrak{X}}(U_i)^n \to \prod \mathcal{O}_{\mathfrak{X}}(U_{ij})^n$ where n is the rank of F. Since U_i is affine, it is clear that $\mathcal{O}_{\mathfrak{X}}(U_i)$ is open and bounded in its generic fibre. The statement therefore follows from the following Lemma. □

Lemma A.3.3. *Let $f : A \to B$ be a morphism of flat $\mathcal{O}_K$-Banach modules. Then if A is open and bounded in its generic fibre, the same is true for the flat $\mathcal{O}_K$-Banach module* $\ker f$.

Proof. Let $M = \ker f$. Then M is a flat $\mathcal{O}_K$-Banach algebra because it is a closed submodule of A. It is open in $M[1/\varpi]$ because it is the preimage of the open $A \subseteq A[1/\varpi]$ under the continuous map $M[1/\varpi] \to A[1/\varpi]$. To see that it is bounded, let $U_0 \subseteq M[1/\varpi]$ be any open subset. Then there is an open $U \subseteq A[1/\varpi]$ such that $U_0 = U \cap M$. Since A is bounded, there is n such that $\varpi^n A \subseteq U$. Since B is ϖ-torsionfree, $U_0 = U \cap M \supseteq (\varpi^n A) \cap M = \varpi^n M$. □

A.3.2 Completion of complexes

Throughout this section we fix a π-adically complete ring R for a non-zero-divisor $\pi \in R$.

One application of Lemma A.3.1 is that it puts us in a situation where we are allowed to commute π-adic completion of a complex of R-modules and cohomology: This is problematic in general, the problem being that the cohomology groups of a complex of complete modules might not be separated. This problem can be remedied by using derived completions, like explained in §2 of [8]. For our applications we will get away with classical operations, due to additional assumptions like bounded π-torsion in the cohomology, as we shall now discuss.

Definition A.3.4. An R-module M has bounded π-torsion if there is $n \in \mathbb{N}$ such that

$$M[\pi^\infty] := \{m \in M | \pi^l m = 0 \text{ for some } l \in \mathbb{N}\} = M[\pi^n].$$

Lemma A.3.5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules all of which have bounded π-torsion. Then the sequence of π-completions $0 \to \hat{A} \to \hat{B} \to \hat{C} \to 0$ is still exact. In particular, for any flat R-module S, also the following sequence is exact:*

$$0 \to A\hat{\otimes}S \to B\hat{\otimes}S \to C\hat{\otimes}S \to 0.$$

Proof. Since C has bounded π-torsion, we have $R\varprojlim_n C[\pi^n] = 0$. Similarly for B. Consequently, in the long exact sequence of inverse systems

$$\mathrm{Tor}_1^R(B, R/\pi^n) = B[\pi^n] \to C[\pi^n] \to A/\pi^n \to B/\pi^n \to C/\pi^n \to 0,$$

all higher derived limits vanish. Thus applying $\varprojlim$ is exact and gives the desired statement.

The statement about S follows since $B \otimes_R S$ and $C \otimes_R S$ still have bounded f-torsion. $\square$

Lemma A.3.6. *Let $C^\bullet$ be a complex of π-torsionfree π-adically complete R-modules. Assume that $H^i(C^\bullet)$ has bounded π-torsion for all $i \in N$. Then for any flat R-module S,*

$$H^i(C^\bullet\hat{\otimes}S) = H^i(C^\bullet)\hat{\otimes}S.$$

Proof. Denote by Z^i, B^i and H^i the cocycles, coboundaries and cohomology of $C^\bullet$.

By the exact sequence $0 \to H^i \to C^i/B^{i-1} \to B^i \to 0$, also C^i/B^{i-1} has bounded π-torsion. Applying Lemma A.3.5 to the exact sequences $0 \to Z^i \to C^i \to B^i \to 0$ and $0 \to B^i \to C^{i+1} \to C^{i+1}/B^i \to 0$ shows that $Z^i\hat{\otimes}S$ and $B^i\hat{\otimes}S$ can be identified with the cycles and boundaries of the complex $C^\bullet\hat{\otimes}S$. Thus $H\hat{\otimes}S = Z^i\hat{\otimes}S/(B^{i-1}\hat{\otimes}S) = H^i(C^\bullet\hat{\otimes}S)$. $\square$

Without assumptions on the torsion to be bounded, while the completion of cohomology might be badly behaved, completion of cocycles does not pose any problems:

Lemma A.3.7. *Let $C^\bullet$ be a complex of π-torsionfree π-adically complete R-algebras. Then $Z^i(C^\bullet)$ is π-adically complete for all i. Moreover, $Z^i(C^\bullet) = \varprojlim_n Z^i(C^\bullet/\pi^n)$.*

Proof. Since $B^{i+1} \subseteq C^{i+1}$ is π-torsionfree, the short exact sequence $0 \to Z^i \to C^i \to B^i \to 0$ stays exact after reducing mod π^n. Comparing this with the exact sequence

$$0 \to Z^i(C^\bullet/\pi^n) \to C^i/\pi^n \to C^{i+1}/\pi^n$$

shows that we have for all n an injection $Z^i/\pi^n \hookrightarrow Z^i(C^\bullet/\pi^n)$. The same sequence shows in the limit that $\varprojlim_n Z^i(C^\bullet/\pi^n)$ is the kernel of $C^i \to C^{i+1}$, which is Z^i. We thus have maps

$$Z^i \to \varprojlim_n Z^i/\pi^n \hookrightarrow \varprojlim_n Z^i(C^\bullet/\pi^n) = Z^i.$$

which all commute with the natural maps from Z^i. In particular, the second arrow is both injective and surjective, thus an isomorphism. This shows that Z^i is π-adically complete. $\square$

A.3.3 Application: completed base-change and left-exact functors

We continue to fix a ring R and a non-zero-divisor $\pi \in R$ such that R is π-adically complete.

Lemma A.3.8.

1. *Let X be a π-torsionfree separated formal scheme over R. Let V be a vector bundle on X. Then $H^0(X, V) = \varprojlim_n H^0(X, V/\pi^n)$ is π-adically complete.*

2. *Assume moreover that R is a discrete valuation ring, S a π-torsionfree π-adically complete R-algebra, and $X_S = X \times_{\mathrm{Spf}(R)} \mathrm{Spf}(S)$ with pullback V_S of V. Then we have*

$$H^0(X_S, V_S) = H^0(X, V)\hat{\otimes}_R S.$$

Proof. Let $C^\bullet$ be the Čech complex of V for a cover of X by affine opens with affine intersections on which V is trivial. Then $C^\bullet/\pi^n$ is the Čech complex of V/π^n and $Z^0(C^\bullet) = H^0(X,V)$ and $Z^0(C^\bullet/\pi^n) = H^0(X,V/\pi^n)$ and the first part follows from Lemma A.3.7.

For part 2, the long exact sequence of sheaf cohomology gives short exact sequences

$$0 \to H^0(X,V)/\pi^n \to H^0(X,V/\pi^n) \to H^1(X,V)[\pi^n] \to 0.$$

Since in the limit the sequence becomes exact by the first part, we have $\varprojlim_n H^1(X,V)[\pi^n] = 0$. When we tensor the above sequence with S, the middle term becomes

$$H^0(X,V/\pi^n)\otimes_R S = \check{H}^0(C^\bullet/\pi^n)\otimes_R S = \check{H}^0(C^\bullet \otimes_R S/\pi^n) = H^0(X,V_S/\pi^n)$$

since S is a flat R-module. We conclude that it suffices to prove $\varprojlim_n (H^1(X,V)[\pi^n]\otimes_R S) = 0$. Since R is a complete discrete valuation ring, this follows from the subsequent Lemma. $\square$

The following two Lemmas are closely related, but different, to the material discussed in Appendix §6.1 of [17], as well as Exposé VII_B in [25].

Lemma A.3.9. *Assume that R is a complete discrete valuation ring with uniformiser π. Let S be a flat π-adically complete R-module and let $(M_i)_{i\in I}$ be an inverse system of π-torsion R-modules such that $\varprojlim_{i\in I} M_i = 0$. Then $\varprojlim_{i\in I}(M_i\otimes_R S) = 0$.*

Proof. We claim that we can always find a family of elements $(v_j)_{j\in J}$ in S for some index set J such that $S = \widehat{\bigoplus}_{j\in J} Rv_j$. If this is true, then since the M_i are π-torsion, we have

$$\varprojlim_{i\in I}(M_i\otimes_R S) = \varprojlim_{i\in I}(M_i\otimes_R R^{\oplus J}) = \varprojlim_{i\in I}\bigoplus_{j\in J} M_i \hookrightarrow \varprojlim_{i\in I}\prod_{j\in J} M_i = \prod_{j\in J}\varprojlim_{i\in I} M_i = 0.$$

To see that one can always find a topological basis of S as claimed, choose a basis $(\overline{v_j})_{i\in J}$ of S/π as an R/π-vector space. Let v_j be any lift to S. Then one sees inductively that $\oplus_J R/\pi^n \to S/\pi^n$, $(a_i)\mapsto \sum a_i v_i$ is an isomorphism, and thus so is the limit $\widehat{\oplus}_J R \to S$. $\square$

Question A.3.10. What are other conditions on R and S for which the Lemma holds? Is the assumption for R to be discrete necessary? When does the Lemma hold for Rlim?

Lemma A.3.11. *Let $A\to B$ be a morphism of π-torsionfree R-algebras. Let G be a profinite group acting continuously on B such that $B^G = A$. Let S be a flat π-adically complete R-algebra. Then we have $(B\hat{\otimes}_R S)^G = A\hat{\otimes}_R S$ in any of the following situations:*

1. *A and B are complete and R is a complete discrete valuation ring, or*
2. *The π-torsion in $H^1(G,B)$ is bounded.*

Proof. For $n\in\mathbb{N}$, the long exact sequence of group cohomology gives a short exact sequence

$$0\to A/\pi^n \to (B/\pi^n)^G \to H^1(G,B)[\pi^n]\to 0$$

Since $-^G$ commutes with $\varprojlim$, the statement $\widehat{B}^G = \widehat{A}$ is equivalent to $\varprojlim_n H^1(G,B)[\pi^n] = 0$, and the desired statement $(B\hat{\otimes}_R S)^G = A\hat{\otimes}_R S$ is equivalent to $\varprojlim_n (H^1(G,B)\otimes_R S)[\pi^n] = 0$.

In case 1, the Lemma therefore follows from Lemma A.3.9.

In case 2, the assumptions that S is a flat R-algebra and that $H^1(G,B)$ has bounded π-torsion combine to show that the π-torsion in $H^1(G,B)\otimes_R S$ is also bounded. But this implies $\varprojlim_n (H^1(G,B)\otimes_R S)[\pi^n] = 0$ as we wanted to see. $\square$

B Table of analogies

Table 1: Table of analogies between perfectoid algebras, perfectoid modular forms, and true modular forms. Here K is a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$ with tilt $K^\flat$ and $t \in \mathcal{O}_{K^\flat}$ such that $\mathcal{O}_{K^\flat}/t = \mathcal{O}_K/p$. Moreover, $(\kappa_n : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times)_{n\in\mathbb{N}}$ and $0 \leq \epsilon \leq \epsilon_\kappa$ are like in Theorem 12.2.1 with corresponding boundary weight $\kappa^\flat : \mathbb{Z}_p^\times \to \mathcal{O}_{K^\flat}^\times$. We set $\kappa := \kappa_0$.

perfectoid K-algebras	K-vector spaces of perfectoid modular forms	K-vector spaces of p-adic modular forms
R	$M_\kappa^{\mathrm{perf}}(\epsilon)$	$M_\kappa(\epsilon)$
R^+	$M_\kappa^{+,\mathrm{perf}}(\epsilon)$	$M_\kappa^+(\epsilon)$
$R^\flat$	$M_{\kappa^\flat}^{\mathrm{perf}}(\epsilon)$	$M_{\kappa^\flat}(\epsilon)$
$R^{\flat+}$	$M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon)$	$M_{\kappa^\flat}^+(\epsilon)$
$X = \mathrm{Spa}(R, R^+)$	$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$	$\mathcal{X}(\epsilon)$
$\mathrm{Spa}(R^\flat, R^{\flat+})$	$\mathcal{X}'(\epsilon)^{\mathrm{perf}}$	$\mathcal{X}'(\epsilon)$
$X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$	$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a^\flat = \mathcal{X}'(\epsilon)^{\mathrm{perf}}$	
$\mathcal{O}_X$	$\omega^{\kappa,\mathrm{perf}}$	ω^κ
$\mathcal{O}_X^+$	$\omega^{\kappa,+,\mathrm{perf}}$	$\omega^{\kappa,+}$
$H^i(X, \mathcal{O}_X^+) \overset{a}{=} 0$	$H^i(\omega^{\kappa,+,\mathrm{perf}}) \overset{a}{=} 0$	
$R^{\flat+}/t \overset{a}{=} R^+/p$	$M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon)/t \overset{a}{=} M_\kappa^{+,\mathrm{perf}}(\epsilon)/p$	$M_{\kappa^\flat}^+(\epsilon)/t \overset{a}{=} M_\kappa^+(\epsilon)/p$
$R^{\flat+} \overset{a}{=} \varprojlim_F R^+$	$M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon) \overset{a}{=} \varprojlim_F M_{\kappa_n^{\flat\sharp}}^{+,\mathrm{perf}}(\epsilon)$	
$R^{\flat+} \overset{a}{=} \varprojlim_F R^+/p$	$M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon) \overset{a}{=} \varprojlim_F M_{\kappa_n}^{+,\mathrm{perf}}(\epsilon)/p$	$M_{\kappa^\flat}^+(\epsilon) \overset{a}{=} \varprojlim M_{\kappa_n}^+(p^{-n}\epsilon)/p$
$W(\mathcal{O}_K)$	perfected weight space $\mathfrak{W}_\infty$	perfected weight space $\mathfrak{W}_\infty$
$W(R^{\flat+})$	families $\mathfrak{f} \in \mathcal{O}(\mathfrak{w}_{\infty,\infty,l})$	families $\mathfrak{f} \in \mathcal{O}(\mathfrak{w}_{r,\infty,l})$
$[-] : R^{\flat+} \to W(R^{\flat+})$	$[-] : M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon) \to \mathcal{O}(\mathfrak{w}_{\infty,\infty,l})$	$[-]_r : M_{\kappa^\flat}^+(\epsilon) \to \mathcal{O}(\mathfrak{w}_{r,\infty,l})$
$\theta : W(R^{\flat+}) \to R^+$	specialis. $\mathfrak{f} \in \mathcal{O}(\mathfrak{w}_{\infty,\infty,l})$ at κ	specialis. $\mathfrak{f} \in \mathcal{O}(\mathfrak{w}_{r,\infty,l})$ at κ
$\sharp : R^{\flat+} \to R^+$	$\sharp : M_{\kappa^\flat}^{+,\mathrm{perf}}(\epsilon) \to M_{\kappa_n}^{+,\mathrm{perf}}(\epsilon)$	$\natural : M_{\kappa^\flat}^+(\epsilon) \to M_{\kappa_n}^+(\epsilon)$
$R^+/p \xrightarrow{F} R^+/p$	$M_{\kappa_{n+1}}^{+,\mathrm{perf}}(\epsilon)/p \xrightarrow[a]{F} M_{\kappa_n}^{+,\mathrm{perf}}(\epsilon)/p$	$M_{\kappa_{n+1}}^+(p^{-n}\epsilon)/p \xrightarrow[a]{F} M_{\kappa_n}^+(\epsilon)/p$

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